STRING VIBRATION

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The lateral displacement u(x, t) of a string is governed by

$$T \frac{\partial^2 u}{\partial x^2} = \rho \frac{\partial^2 u}{\partial t^2} \tag{1}$$

where

- ρ is the mass per unit length,
- T is the tension.

Furthermore, the string has length L.

Equation (1) is taken from Reference 1. It assumes that the lateral displacement is small so that T is constant.

$$\frac{\partial^2 u}{\partial x^2} = \left(\frac{\rho}{T}\right)\frac{\partial^2 u}{\partial t^2} \tag{2}$$

Let

$$c = \sqrt{\frac{T}{\rho}} \tag{3}$$

Note that c is the wave propagation velocity. Substitute equation (3) into (2).

$$\frac{\partial^2 u}{\partial x^2} = \left(\frac{1}{c^2}\right)\frac{\partial^2 u}{\partial t^2} \tag{4}$$

Separate the variables. Let

$$u(x,t) = U(x)G(t) \tag{5}$$

Substitute equation (4) into (3).

$$\frac{\partial^2}{\partial x^2}\left[U(x)G(t)\right] = \left(\frac{1}{c^2}\right)\frac{\partial^2}{\partial t^2}\left[U(x)G(t)\right] \tag{6}$$

Perform the partial differentiation.

$$U''(x)G(t) = \left(\frac{1}{c^2}\right)U(x)G''(t) \tag{7}$$

Divide through by $U(x)G(t)$.

$$\frac{U''(x)}{U(x)} = \left(\frac{1}{c^2}\right)\frac{G''(t)}{G(t)} \tag{8}$$

$$c^2\,\frac{U''(x)}{U(x)} = \frac{G''(t)}{G(t)} \tag{9}$$

Each side of equation (9) must equal a constant. Let ω be a constant.

$$c^2\,\frac{U''(x)}{U(x)} = \frac{G''(t)}{G(t)} = -\omega^2 \tag{10}$$

The time equation is

$$\frac{G''(t)}{G(t)} = -\omega^2 \tag{11}$$

$$G''(t) = -\omega^2 G(t) \tag{12}$$

$$G''(t) + \omega^2 G(t) = 0 \tag{13}$$

Propose a solution

$$G(t) = a\sin(\omega t) + b\cos(\omega t) \tag{14}$$

$$G'(t) = a\omega\cos(\omega t) - b\omega\sin(\omega t) \tag{15}$$

$$G''(t) = -a\omega^2\sin(\omega t) - b\omega^2\cos(\omega t) \tag{16}$$

Verify the proposed solution. Substitute into equation (13).

$$-a\,\omega^2\sin(\omega t) - b\,\omega^2\cos(\omega t) + \omega^2\left[\sin(\omega t) + \omega^2\cos(\omega t)\right] = 0 \tag{17}$$

$$0 = 0 \tag{18}$$

Equation (14) is thus verified as a solution.

There is not a unique ω, however, in equation (10). This is demonstrated later in the derivation. Thus a subscript n must be added as follows.

$$G_n(t) = a_n \sin(\omega_n t) + b_n \cos(\omega_n t) \tag{19}$$

The spatial equation is

$$c^2 \frac{U''(x)}{U(x)} = -\omega^2 \tag{20}$$

$$c^2 U''(x) = -\omega^2 U(x) \tag{21}$$

$$c^2 U''(x) + \omega^2 U(x) = 0 \tag{22}$$

$$U''(x) + \frac{\omega^2}{c^2} U(x) = 0 \tag{23}$$

Equation (23) is similar to equation (13). Thus, a solution can be found by inspection.

$$U(x) = d \sin\left(\frac{\omega x}{c}\right) + e \cos\left(\frac{\omega x}{c}\right) \tag{24}$$

The slope equation is

$$U'(x) = \left[\frac{\omega}{c}\right]\left[d \cos\left(\frac{\omega x}{c}\right) - e \sin\left(\frac{\omega x}{c}\right)\right] \tag{25}$$

Now consider boundary condition cases.

<u>Case I. Fixed-Fixed</u>

The left boundary condition is

$$u(0,t) = 0 \qquad \text{(zero displacement)} \tag{26}$$

$$U(0)G(t) = 0 \tag{27}$$

$$U(0) = 0 \tag{28}$$

The right boundary condition is

$$u(L,t) = 0 \qquad \text{(zero displacement)} \tag{29}$$

$$U(L)G(t) = 0 \tag{30}$$

$$U(L) = 0 \tag{31}$$

Substitute equation (28) into (24).

$$e = 0 \tag{32}$$

Thus, the displacement equation becomes

$$U(x) = d \sin\left(\frac{\omega x}{c}\right) \tag{33}$$

Substitute equation (31) into (33).

$$d \sin\left(\frac{\omega L}{c}\right) = 0 \tag{34}$$

The constant d must be non-zero for a non-trivial solution. Thus,

$$\frac{\omega_n L}{c} = n\pi, \quad n = 1, 2, 3, \ldots \tag{35}$$

The ω term is given a subscript n because there are multiple roots.

$$\omega_n = n\pi \frac{c}{L}, \quad n = 1, 2, 3, \ldots \tag{36a}$$

The natural frequency f_n is given by

$$f_n = \frac{nc}{2L}, \quad n = 1, 2, 3, \ldots \tag{36b}$$

Typically,

ω is in units of radians/sec,

f_n is in units of Hertz.

The displacement function the fixed-fixed string is

$$U_n(x) = d_n \sin\left(\frac{\omega_n x}{c}\right) \tag{37}$$

$$U_n(x) = d_n \sin\left(\frac{n\pi x}{L}\right) \tag{38}$$

Substitute the natural frequency term into the time equation.

$$G_n(t) = a_n \sin\left(\frac{n\pi ct}{L}\right) + b_n \cos\left(\frac{n\pi ct}{L}\right) \tag{39}$$

The displacement function is thus

$$u(x,t) = \sum_{n=1}^{\infty}\left\{\left[d_n \sin\left(\frac{n\pi x}{L}\right)\right]\left[a_n \sin\left(\frac{n\pi c t}{L}\right) + b_n \cos\left(\frac{n\pi c t}{L}\right)\right]\right\} \tag{40}$$

The coefficients can be simplified as follows

$$A_n = d_n a_n \tag{41}$$

$$B_n = d_n b_n \tag{42}$$

By substitution, the displacement equation is

$$u(x,t) = \sum_{n=1}^{\infty}\left\{\left[\sin\left(\frac{n\pi x}{L}\right)\right]\left[A_n \sin\left(\frac{n\pi c t}{L}\right) + B_n \cos\left(\frac{n\pi c t}{L}\right)\right]\right\} \tag{43a}$$

An equivalent form is

$$u(x,t) = \sum_{n=1}^{\infty}\{[\sin(k_n x)][A_n \sin(\omega_n t) + B_n \cos(\omega_n t)]\} \tag{43b}$$

where

$$k_n = \left(\frac{n\pi}{L}\right) \tag{43c}$$

The velocity equation is

$$\frac{\partial}{\partial t} u(x,t) = \sum_{n=1}^{\infty}\left\{\left(\frac{n\pi c}{L}\right)\left[\sin\left(\frac{n\pi x}{L}\right)\right]\left[A_n \cos\left(\frac{n\pi ct}{L}\right) - B_n \sin\left(\frac{n\pi ct}{L}\right)\right]\right\} \tag{44a}$$

The equivalent form is

$$\frac{\partial}{\partial t}u(x,t) = \sum_{n=1}^{\infty}\{(\omega_n)[\sin(k_n x)][A_n \cos(\omega_n t) - B_n \sin(\omega_n t)]\} \tag{44b}$$

Now consider that the string is given an initial displacement of u(x,0) and initial velocity of v(x,0).

Solve for the coefficients in equation (43).

The initial displacement equation is

$$u(x,0) = \sum_{n=1}^{\infty}\left\{B_n \sin\left(\frac{n\pi x}{L}\right)\right\} \tag{45}$$

Multiply each side by $\sin\left(\frac{m\pi x}{L}\right)$. Then integrate with respect to x, from 0 to L.

$$\int_0^L u(x,0)\sin\left(\frac{m\pi x}{L}\right)dx = \int_0^L \left\{\sum_{n=1}^{\infty}\left\{B_n \sin\left(\frac{n\pi x}{L}\right)\right\}\sin\left(\frac{m\pi x}{L}\right)\right\}dx \tag{46}$$

Now consider the case where n = m,

$$\int_0^L u(x,0)\sin\left(\frac{m\pi x}{L}\right)dx = \int_0^L \left\{B_m \sin^2\left(\frac{m\pi x}{L}\right)\right\}dx \tag{47}$$

$$\int_0^L u(x,0)\sin\left(\frac{m\pi x}{L}\right)dx = \frac{1}{2}\int_0^L \left\{B_m\left[1-\cos\left(\frac{2m\pi x}{L}\right)\right]\right\}dx \tag{48}$$

$$\int_0^L u(x,0)\sin\left(\frac{m\pi x}{L}\right)dx = \frac{1}{2}\left\{B_m\left[x-\left(\frac{L}{2m\pi}\right)\sin\left(\frac{2m\pi x}{L}\right)\right]\right\}\Bigg|_0^L \tag{49}$$

$$\int_0^L u(x,0)\sin\left(\frac{m\pi x}{L}\right)dx = \frac{1}{2}B_m L \tag{50}$$

$$B_m = \frac{2}{L}\int_0^L u(x,0)\sin\left(\frac{m\pi x}{L}\right)dx \tag{51}$$

Now consider $n \neq m$,

$$\int_0^L u(x,0)\sin\left(\frac{m\pi x}{L}\right)dx = \sum_{n=1}^{\infty}\left\{\frac{1}{2}B_n\int_0^L\left[\cos\left(\frac{(n-m)\pi x}{L}\right)-\cos\left(\frac{(n+m)\pi x}{L}\right)\right]dx\right\} \tag{52}$$

$$\int_0^L u(x,0)\sin\left(\frac{m\pi x}{L}\right)dx$$

$$= \sum_{n=1}^{\infty}\left\{\frac{1}{2}B_n\left[\left(\frac{L}{(n-m)\pi}\right)\sin\left(\frac{(n-m)\pi x}{L}\right)-\left(\frac{L}{(n+m)\pi}\right)\sin\left(\frac{(n+m)\pi x}{L}\right)\right]\Bigg|_0^L\right\} \tag{53}$$

$$\int_0^L u(x,0)\sin\left(\frac{m\pi x}{L}\right)dx = 0 \qquad \text{for } n \neq m \tag{54}$$

The left-hand side may be nonzero, however. Thus, n must equal m, as shown in equations (47) through (51).

The initial velocity equation is

$$v(x, 0) = \sum_{n=1}^{\infty}\left\{A_n\left(\frac{L}{n\pi}\right)\sin\left(\frac{n\pi x}{L}\right)\right\} \tag{55}$$

The initial velocity equation (55) is very similar to the initial displacement equation (45). Thus the solution can be written by inspection.

$$A_m = \frac{2}{m\pi}\int_0^L v(x,0)\sin\left(\frac{m\pi x}{L}\right)dx \tag{56}$$

REFERENCE

1. W. Thomson, Theory of Vibration with Applications, Second Edition, Prentice-Hall, New Jersey, 1981.