LONGITUDINAL VIBRATION OF A TAPERED ROD

By Tom Irvine
Email: tomirvine@aol.com

October 5, 2003

Introduction

Consider a thin, tapered rod.

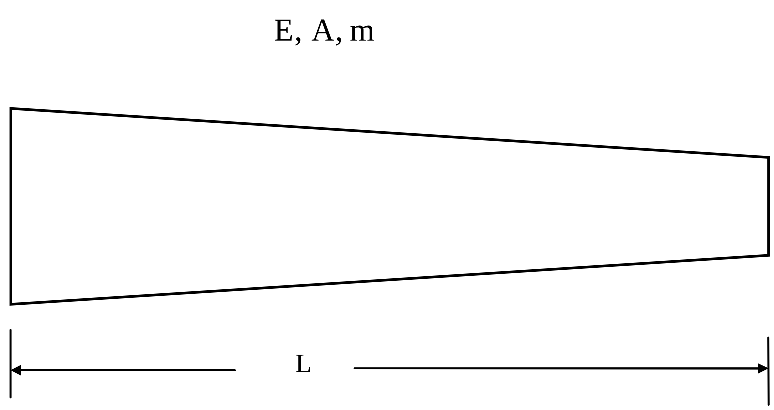

Figure 1.

E	is the modulus of elasticity
L	is the length
A_0	is the cross-section area at x = 0
A_L	is the cross-section area at x = L
m	is the mass per length
ρ	is the mass per volume

The product of the elastic modulus and area is

$$EA(x) = E\left\{A_0\left[1-\frac{x}{L}\right]+A_L\left[\frac{x}{L}\right]\right\} \quad (1)$$

$$EA(x) = E\left\{A_0+\frac{x}{L}\left(A_L-A_0\right)\right\} \quad (2)$$

$$EA(x) = EA_0\left\{1+\frac{x}{L}\left(\frac{A_L}{A_0}-1\right)\right\} \quad (3)$$

Let

$$\alpha = \frac{A_L}{A_0}-1 \quad (4)$$

$$EA(x) = EA_0\left\{1+\alpha\frac{x}{L}\right\} \quad (5)$$

Similarly

$$m(x) = \rho A_0\left\{1+\alpha\frac{x}{L}\right\} \quad (6)$$

The longitudinal displacement u(x, t) is governed by the equation

$$\frac{\partial}{\partial x}\left[EA(x)\frac{\partial u}{\partial x}\right] = m(x)\frac{\partial^2 u}{\partial t^2} \quad (7)$$

This equation is taken from Reference 1.

Separate the variables. Let

$$u(x,t) = U(x)T(t) \quad (8)$$

Substitute equation (8) into (7).

$$\frac{\partial}{\partial x}\left[EA(x)\frac{\partial}{\partial x}\left[U(x)T(t)\right]\right]=\frac{\partial^2}{\partial t^2}\left[m(x)U(x)T(t)\right] \tag{9}$$

Perform the partial differentiation.

$$T(t)\,\frac{\partial}{\partial x}\left[EA(x)\frac{\partial}{\partial x}\left[U(x)T(t)\right]\right]=\left[m(x)U(x)\right]\frac{\partial^2}{\partial t^2}T(t) \tag{10}$$

Divide through by U(x)T(t).

$$\frac{1}{\left[m(x)U(x)\right]}\frac{\partial}{\partial x}\left[EA(x)\frac{\partial}{\partial x}U(x)\right]=\frac{1}{T(t)}\frac{\partial^2}{\partial t^2}T(t) \tag{11}$$

Each side of equation (11) must equal a constant. Let ω be a constant.

$$\frac{1}{\left[m(x)U(x)\right]}\frac{\partial}{\partial x}\left[EA(x)\frac{\partial}{\partial x}U(x)\right]=\frac{1}{T(t)}\frac{\partial^2}{\partial t^2}T(t)\ =-\omega^2 \tag{12}$$

Change the partial derivatives to ordinary derivatives

$$\frac{1}{\left[m(x)U(x)\right]}\,\frac{d}{dx}\left[EA(x)\frac{d}{dx}U(x)\right]=\frac{1}{T(t)}\frac{d^2}{dt^2}T(t)\ =-\omega^2 \tag{13}$$

The spatial equation is

$$\frac{1}{\left[m(x)U(x)\right]}\,\frac{d}{dx}\left[EA(x)\frac{d}{dx}U(x)\right]\ =-\omega^2 \tag{14}$$

$$\frac{1}{T(t)}\frac{d^2}{dt^2}T(t) = -\omega^2 \tag{15}$$

$$\frac{d}{dx}\left[EA(x)\frac{d}{dx}U(x)\right] + \omega^2 m(x)U(x) = 0 \tag{16}$$

$$\left[EA(x)\frac{d^2}{dx^2}U(x)\right] + \left[\frac{d}{dx}EA(x)\right]\left[\frac{d}{dx}U(x)\right] + \omega^2 m(x)U(x) = 0 \tag{17}$$

Recall

$$EA(x) = EA_0\left\{1 + \alpha\frac{x}{L}\right\} \tag{18}$$

$$m(x) = \rho\, A_0\left\{1 + \alpha\frac{x}{L}\right\} \tag{19}$$

$$EA_0\left\{1 + \alpha\frac{x}{L}\right\}\frac{d^2}{dx^2}U(x) + \left[\frac{d}{dx}EA_0\left\{1 + \alpha\frac{x}{L}\right\}\right]\left[\frac{d}{dx}U(x)\right]$$
$$+ \omega^2\rho\, A_0\left\{1 + \alpha\frac{x}{L}\right\}U(x) = 0 \tag{20}$$

$$E\left\{1 + \alpha\frac{x}{L}\right\}\frac{d^2}{dx^2}U(x) + E\left[\frac{d}{dx}\left\{1 + \alpha\frac{x}{L}\right\}\right]\left[\frac{d}{dx}U(x)\right]$$
$$+ \omega^2\rho\left\{1 + \alpha\frac{x}{L}\right\}U(x) = 0 \tag{21}$$

$$\left\{1 + \alpha \frac{x}{L}\right\}\frac{d^2}{dx^2}U(x) + \left[\frac{d}{dx}\left\{1 + \alpha \frac{x}{L}\right\}\right]\left[\frac{d}{dx}U(x)\right]$$

$$+ \frac{\omega^2}{c^2}\left\{1 + \alpha \frac{x}{L}\right\}U(x) = 0 \tag{22}$$

$$\left\{1 + \alpha \frac{x}{L}\right\}\frac{d^2}{dx^2}U(x) + \frac{\alpha}{L}\left[\frac{d}{dx}U(x)\right] + \frac{\omega^2}{c^2}\left\{1 + \alpha \frac{x}{L}\right\}U(x) = 0 \tag{23}$$

Let

$$v = \left\{1 + \alpha \frac{x}{L}\right\} \tag{24}$$

$$dv = \frac{\alpha}{L}dx \tag{25}$$

$$\frac{dv}{dx} = \frac{\alpha}{L} \tag{26}$$

$$\frac{du}{dx} = \frac{dv}{dx}\frac{dU}{dv} \tag{27}$$

$$\frac{d}{dx} = \frac{dv}{dx}\frac{d}{dv} \tag{28}$$

$$\frac{d}{dx} = \frac{\alpha}{L}\frac{d}{dv} \tag{29}$$

$$\frac{dU}{dx} = \frac{\alpha}{L}\frac{dU}{dv} \tag{30}$$

$$\frac{d^2U}{dx^2} = \frac{\alpha}{L}\frac{d}{dv}\left[\frac{\alpha}{L}\frac{dU}{dv}\right] \tag{31}$$

$$\frac{d^2U}{dx^2} = \frac{\alpha^2}{L^2}\frac{d^2U}{dv^2} \tag{32}$$

$$\frac{\alpha^2}{L^2}\, v\, \frac{d^2}{dv^2} U \;+\; \frac{\alpha^2}{L^2}\frac{d}{dv} U \;+\; \frac{\omega^2}{c^2}\; vU = 0 \tag{33}$$

$$v\, \frac{d^2}{dv^2} U \;+\; \frac{d}{dv} U \;+\; \left(\frac{\omega L}{c\alpha}\right)^2\; vU = 0 \tag{34}$$

$$v^2\, \frac{d^2}{dv^2} U \;+\; v\frac{d}{dv} U \;+\; \left(\frac{\omega L}{c\alpha}\right)^2\; v^2 U = 0 \tag{35}$$

$$\beta = \left(\frac{\omega L}{c\alpha}\right) \tag{36}$$

$$v^2\, \frac{d^2}{dv^2} U \;+\; v\frac{d}{dv} U \;+\; \beta^2\; v^2 U = 0 \tag{37}$$

$$v^2\, \frac{d^2}{dv^2} U \;+\; v\frac{d}{dv} U \;+\; (\beta\, v)^2\; U = 0 \tag{38}$$

$$v^2\, \frac{d^2}{dv^2} U \;+\; v\frac{d}{dv} U \;+\; (\beta\, v)^2\; U = 0 \tag{39}$$

Let

$$z = \beta v \tag{40}$$

$$v = \frac{z}{\beta} \tag{41}$$

$$dv = \frac{1}{\beta} dz \tag{42}$$

$$\frac{dv}{dz} = \frac{1}{\beta} \tag{43}$$

$$\frac{dz}{dv} = \beta \tag{44}$$

$$\frac{d}{dv} U = \frac{dz}{dv} \frac{dU}{dz} \tag{45}$$

$$\frac{d}{dv} U = \beta \frac{dU}{dz} \tag{46}$$

$$\frac{d^2}{dv^2} U = \frac{dz}{dv} \frac{d}{dz} \left[\frac{dz}{dv} \frac{d}{dz} U \right] \tag{47}$$

$$\frac{d^2}{dv^2} U = \beta \frac{d}{dz} \left[\beta \frac{d}{dz} U \right] \tag{48}$$

$$\frac{d^2}{dv^2} U = \beta^2 \frac{d^2 U}{dz^2} \tag{49}$$

$$\frac{z^2}{\beta^2}\beta^2\frac{d^2U}{dz^2} + \frac{z}{\beta}\beta\frac{dU}{dz} + z^2\ U = 0 \tag{50}$$

$$z^2\ \frac{d^2U}{dz^2} + z\frac{dU}{dz} + z^2\ U = 0 \tag{51}$$

The solution is a Bessel function of order zero.

$$U(z) = B_1\ J_0(z) + B_2\ Y_0(z) \tag{52}$$

$$\frac{d}{dz}U(z) = \frac{1}{2}B_1\left[J_{-1}(z) - J_1(z)\right] + \frac{1}{2}B_2\left[Y_{-1}(z) - Y_1(z)\right] \tag{53}$$

B_1 and B_2 are constant coefficients.

The Bessel functions are

$$J_o(x) = 1 - \frac{(x/2)^2}{(1!)^2} + \frac{(x/2)^4}{(2!)^2} - \frac{(x/2)^6}{(3!)^2} + \ldots \tag{54}$$

$$Y_o(x) = \frac{2}{\pi}\left[\left(\ln\frac{x}{2} + C\right)J_o(x) + \frac{2}{1}J_2(x) - \frac{2}{2}J_4(x) + \frac{2}{3}J_6(x) - \ldots\right] \tag{55}$$

where $C = 0.577\ 215\ 665$

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{2k+n}}{k!\,\Gamma(k+1+n)} \tag{56}$$

$$\Gamma(x) = \lim_{n \to \infty} \left\{ \frac{n^x\, n!}{x(x+1)(x+2)\ldots(x+n)} \right\} \tag{57}$$

The eigenvalues are calculated by applying the boundary conditions. A special case is considered in Appendix A.

APPENDIX A

Special Case: Fixed-Free Rod with $A_L = 0$

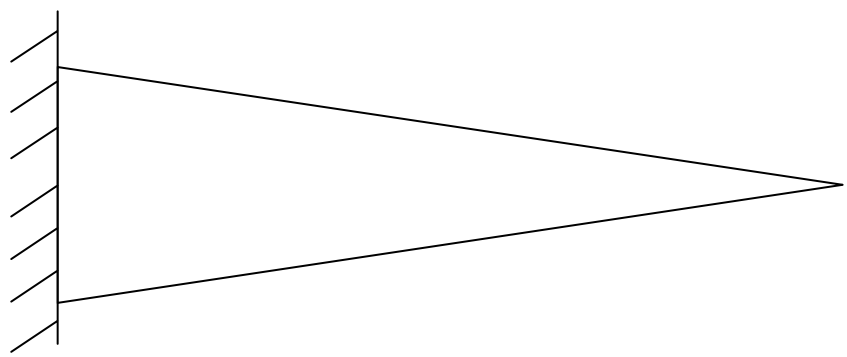

Figure A-1.

$$\alpha = \frac{A_L}{A_0} - 1 \qquad \text{(A-1)}$$

$$A_L = 0 \qquad \text{(A-2)}$$

$$\alpha = -1 \qquad \text{(A-3)}$$

$$U(z) = B_1 J_0(z) + B_2 Y_0(z) \qquad \text{(A-4)}$$

$$\frac{d}{dz} U(z) = \frac{1}{2} B_1 \left[J_{-1}(z) - J_1(z) \right] + \frac{1}{2} B_2 \left[Y_{-1}(z) - Y_1(z) \right] \qquad \text{(A-5)}$$

$$z = \beta v \tag{A-6}$$

$$z = -\left(\frac{\omega L}{c} v\right) \tag{A-7}$$

$$z = -\frac{\omega L}{c}\left\{1 - \frac{x}{L}\right\} \tag{A-8}$$

Consider a fixed-free rod such that

$$U(x=0) = 0 \tag{A-9}$$

$$\frac{d}{dx} U \Big|_{x=L} = 0 \tag{A-10}$$

$$z(x=0) = -\frac{\omega L}{c} \tag{A-11}$$

$$U(x=0) = B_1 J_0\left(-\frac{\omega L}{c}\right) + B_2 Y_0\left(-\frac{\omega L}{c}\right) \tag{A-12}$$

Note that

$$\lim_{x \to 0} Y_0(x) \to -\infty \tag{A-13}$$

Furthermore, $Y_0(x)$ is undefined for $x \le 0$.

Thus,

$$B_2 = 0 \tag{A-14}$$

$$U(z) = B_1 \, J_0(z) \tag{A-15}$$

$$\frac{d}{dz} U(z) = \frac{1}{2} B_1 \left[J_{-1}(z) - J_1(z) \right] \tag{A-16}$$

$$J_{-1}(z) = -2 \, J_1(z) \tag{A-17}$$

$$\frac{d}{dz} U(z) = -B_1 \, J_1(z) \tag{A-18}$$

$$U(x = 0) = B_1 J_0 \left(-\frac{\omega L}{c} \right) \tag{A-19}$$

$$B_1 \neq 0 \tag{A-20}$$

$$J_0 \left(-\frac{\omega L}{c} \right) = 0 \tag{A-21}$$

$$J_0(x) = J_0(-x) \tag{A-22}$$

$$J_0 \left(\frac{\omega L}{c} \right) = 0 \tag{A-23}$$

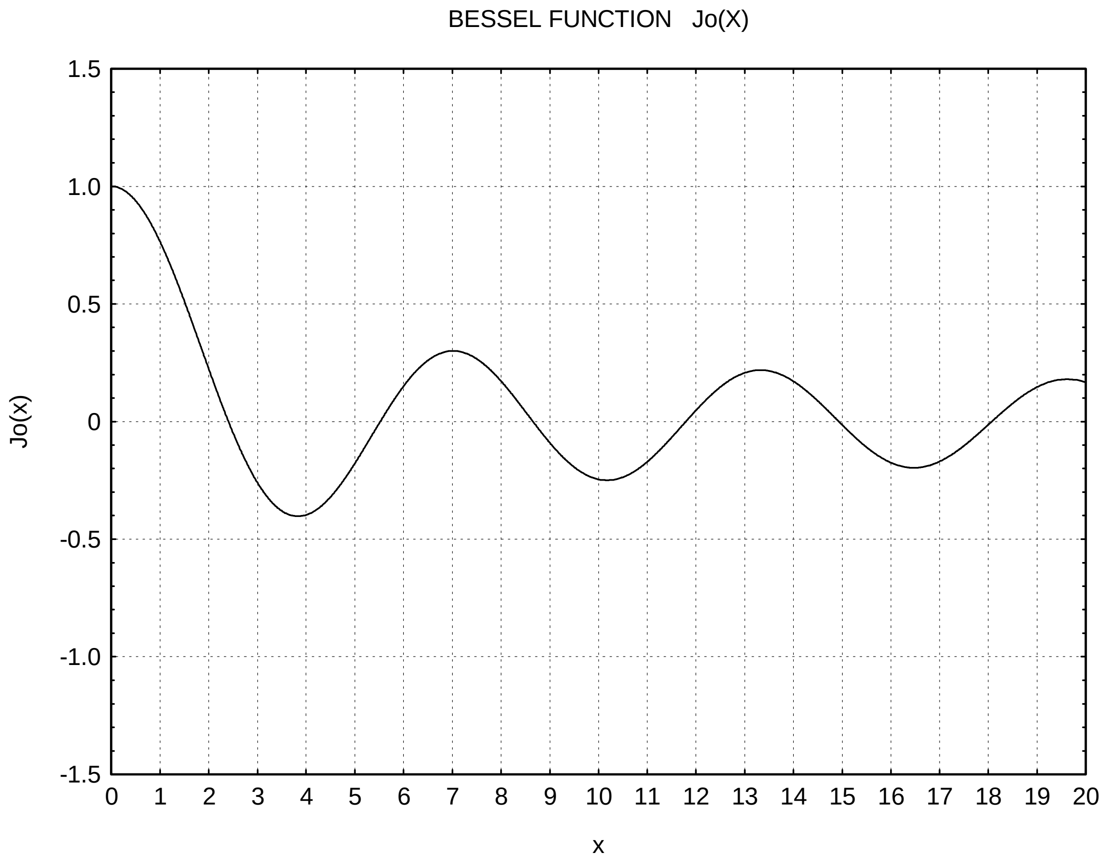

Figure A-1.

Table A-1. Roots	
Root	Value
1	2.404826
2	5.520078
3	8.653728

Thus,

$$\omega_1 = \quad 2.404826 \ \frac{c}{L} \qquad \text{(A-24)}$$