LONGITUDINAL VIBRATION OF A TAPERED ROD VIA THE FINITE ELEMENT METHOD

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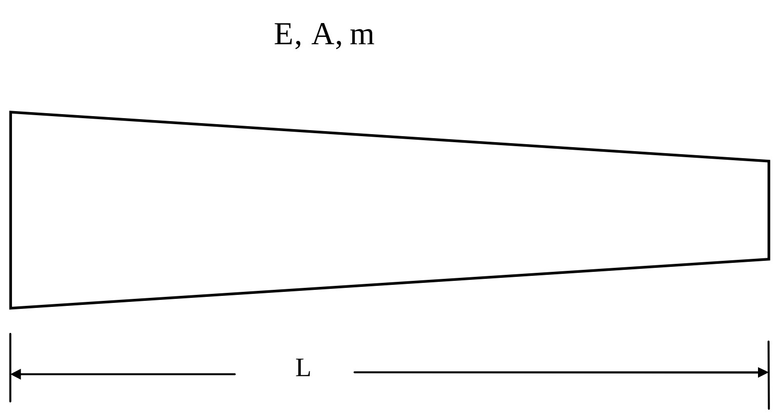
Introduction

Consider a thin, tapered rod.



E	is the modulus of elasticity.
L	is the length.
A	is the cross-section area.
m	is the mass per length.
ρ	is the mass per volume.
A_0	is the area at x = 0
A_L	is the area at x = L

The stiffness term is

$$EA(x) = E\left\{A_0\left[1-\frac{x}{L}\right]+A_L\left[\frac{x}{L}\right]\right\} \tag{1}$$

$$EA(x) = E\left\{A_0+\frac{x}{L}\left(A_L - A_0\right)\right\} \tag{2}$$

$$EA(x) = EA_0\left\{1+\frac{x}{L}\left(\frac{A_L}{A_0}-1\right)\right\} \tag{3}$$

Let

$$\alpha = \frac{A_L}{A_0}-1$$

$$EA(x) = EA_0\left\{1+\alpha\,\frac{x}{L}\right\} \tag{4}$$

Similarly

$$m(x) = \rho\, A_0\left\{1+\alpha\,\frac{x}{L}\right\} \tag{5}$$

The longitudinal displacement u(x, t) is governed by the equation

$$\frac{\partial}{\partial x}\left[EA(x)\frac{\partial u}{\partial x}\right] = m(x)\frac{\partial^2 u}{\partial t^2} \tag{6}$$

This equation is taken from Reference 1.

Separate the variables. Let

$$u(x,t) = U(x)T(t) \tag{7}$$

Substitute equation (7) into (6).

$$\frac{\partial}{\partial x}\left[EA(x)\frac{\partial}{\partial x}\left[U(x)T(t)\right]\right]=\frac{\partial^2}{\partial t^2}\left[m(x)U(x)T(t)\right] \tag{8}$$

Perform the partial differentiation.

$$T(t)\ \frac{\partial}{\partial x}\left[EA(x)\frac{\partial}{\partial x}\left[U(x)T(t)\right]\right]=\left[m(x)U(x)\right]\frac{\partial^2}{\partial t^2}T(t) \tag{9}$$

Divide through by U(x)T(t).

$$\frac{1}{\left[m(x)U(x)\right]}\frac{\partial}{\partial x}\left[EA(x)\frac{\partial}{\partial x}U(x)\right]=\frac{1}{T(t)}\frac{\partial^2}{\partial t^2}T(t) \tag{10}$$

Each side of equation (7) must equal a constant. Let ω be a constant.

$$\frac{1}{\left[m(x)U(x)\right]}\frac{\partial}{\partial x}\left[EA(x)\frac{\partial}{\partial x}U(x)\right]=\frac{1}{T(t)}\frac{\partial^2}{\partial t^2}T(t)\ =-\omega^2 \tag{11}$$

Change the partial derivatives to ordinary derivatives

$$\frac{1}{\left[m(x)U(x)\right]}\ \frac{d}{dx}\left[EA(x)\frac{d}{dx}U(x)\right]=\frac{1}{T(t)}\frac{d^2}{dt^2}T(t)\ =-\omega^2 \tag{12}$$

The spatial equation is

$$\frac{1}{\left[m(x)U(x)\right]}\ \frac{d}{dx}\left[EA(x)\frac{d}{dx}U(x)\right]\ =-\omega^2 \tag{13}$$

$$\frac{1}{T(t)}\frac{d^2}{dt^2}T(t)\ =-\omega^2 \tag{14}$$

$$\frac{d}{dx}\left[EA(x)\frac{d}{dx}U(x) \right] + \omega^2 m(x)U(x) = 0 \tag{15}$$

$$\left[EA(x)\frac{d^2}{dx^2}U(x) \right] + \left[\frac{d}{dx}EA(x) \right]\left[\frac{d}{dx}U(x) \right] + \omega^2 m(x)U(x) = 0 \tag{16}$$

The weighted residual method is applied to equation (16). This method is suitable for boundary value problems.

There are numerous techniques for applying the weighted residual method. Specifically, the Galerkin approach is used in this tutorial.

The differential equation (16) is multiplied by a test function $\phi(x)$. Note that the test function $\phi(x)$ must satisfy the homogeneous essential boundary conditions. The essential boundary conditions are the prescribed values of U and its first derivative.

The test function is not required to satisfy the differential equation, however.

The product of the test function and the differential equation is integrated over the domain. The integral is set equation to zero.

$$\int \phi(x) \left\{ \left[EA(x)\frac{d^2}{dx^2}U(x) \right] + \left[\frac{d}{dx}EA(x) \right]\left[\frac{d}{dx}U(x) \right] + \omega^2 m(x)U(x) \right\} dx = 0 \tag{17}$$

The test function $\phi(x)$ can be regarded as a virtual displacement. The differential equation in the brackets represents an internal force. This term is also regarded as the residual. Thus, the integral represents virtual work, which should vanish at the equilibrium condition.

Define the domain over the limits from a to b. These limits represent the boundary points of the entire rod.

$$\int_a^b \phi(x)\left\{\left[EA(x)\frac{d^2}{dx^2}U(x)\right] + \left[\frac{d}{dx}EA(x)\right]\left[\frac{d}{dx}U(x)\right] + \omega^2 m(x)U(x)\right\}dx = 0$$

(18)

$$\int_a^b \phi(x)\left\{\left[EA(x)\frac{d^2}{dx^2}U(x)\right]\right\}dx + \int_a^b \phi(x)\left\{\left[\frac{d}{dx}EA(x)\right]\left[\frac{d}{dx}U(x)\right]\right\}dx$$
$$+\int_a^b \phi(x)\left\{\omega^2 m(x)U(x)\right\}dx = 0$$

(19)

Integrate the first integral by parts.

$$\int_a^b \frac{d}{dx}\left\{\phi(x)\left[EA(x)\frac{d}{dx}U(x)\right]\right\}dx$$

$$-\int_a^b \left\{\phi(x)\left[\frac{d}{dx}EA(x)\right]\left[\frac{d}{dx}U(x)\right]\right\}dx - \int_a^b \left\{EA(x)\left[\frac{d}{dx}\phi(x)\right]\left[\frac{d}{dx}U(x)\right]\right\}dx$$

$$+ \int_a^b \phi(x)\left\{\left[\frac{d}{dx}EA(x)\right]\left[\frac{d}{dx}U(x)\right]\right\}dx$$

$$+\int_a^b \phi(x)\left\{\omega^2 m(x)U(x)\right\}dx = 0$$

(20)

$$\int_a^b \frac{d}{dx}\left\{\phi(x)\left[EA(x)\frac{d}{dx}U(x)\right]\right\}dx - \int_a^b \left\{EA(x)\left[\frac{d}{dx}\phi(x)\right]\left[\frac{d}{dx}U(x)\right]\right\}dx$$

$$+\int_a^b \phi(x)\left\{\omega^2 m(x)U(x)\right\}dx = 0 \tag{21}$$

$$\phi(x)\left[EA(x)\frac{d}{dx}U(x)\right]\Bigg|_a^b - \int_a^b \left\{EA(x)\left[\frac{d}{dx}\phi(x)\right]\left[\frac{d}{dx}U(x)\right]\right\}dx$$

$$+\int_a^b \phi(x)\left\{\omega^2 m(x)U(x)\right\}dx = 0 \tag{22}$$

Consider a fixed-free rod. The boundary conditions are

$$U(a) = 0 \tag{23}$$

$$EA(x)\frac{d}{dx}U(x)\Bigg|_{x=b} = 0 \tag{24}$$

Thus, the test functions must satisfy

$$\phi(a) = 0 \tag{25}$$

The boundary conditions require

$$\left\{\phi(x)\left[EA(x)\frac{d}{dx}U(x)\right]\right\}\Bigg|_a^b = 0 \tag{26}$$

Apply the boundary conditions to equation (26). The result is

$$-\int_a^b \left\{ EA(x)\left[\frac{d}{dx}\phi(x)\right]\left[\frac{d}{dx}U(x)\right] \right\} dx \quad + \quad \omega^2 \int_a^b \phi(x)\left\{ m(x)U(x) \right\} dx = 0 \tag{27}$$

Note that equation (27) would also be obtained for other simple boundary condition cases.

Now consider that the rod consists of number of segments, or elements. The elements are arranged geometrically in series form.

Furthermore, the endpoints of each element are called nodes.

The following equation must be satisfied for each element.

$$-\int \left\{ EA(x)\left[\frac{d}{dx}\phi(x)\right]\left[\frac{d}{dx}U(x)\right] \right\} dx \quad + \quad \omega^2 \int \phi(x)\left\{ m(x)U(x) \right\} dx = 0 \tag{28}$$

$$-\int \left\{ EA_0\left[1+\alpha\frac{x}{L}\right]\left[\frac{d}{dx}\phi(x)\right]\left[\frac{d}{dx}U(x)\right] \right\} dx$$
$$+ \quad \omega^2 \int \phi(x)\left\{ \rho\, A_0\left[1+\alpha\frac{x}{L}\right] U(x) \right\} dx = 0 \tag{29}$$

$$-\,EA_0 \int \left\{ \left[1+\alpha\frac{x}{L}\right]\left[\frac{d}{dx}\phi(x)\right]\left[\frac{d}{dx}U(x)\right] \right\} dx$$
$$+ \quad \rho\, A_0\, \omega^2 \int \left\{ \left[1+\alpha\frac{x}{L}\right] \phi(x)U(x) \right\} dx = 0 \tag{30}$$

Furthermore, consider that the stiffness and mass properties are constant for a given element. Express the displacement function U(x) in terms of nodal displacements u_{j-1} and u_j .

$$U(x) = L_1\, u_{j-1} + L_2\, u_j \ , \qquad (j-1)h \ \le x \le jh \tag{31}$$

Note that h is the element length. In addition, each L coefficients is a function of x.

Now introduce a nondimensional natural coordinate ξ. The variable j is the element number, j = 0, 1, … , n.

$$\xi = j - x/h \tag{32}$$

$$h\,\xi = h\,j - x \tag{33}$$

$$x = h\,j - h\,\xi \tag{34}$$

$$\left(\frac{x}{h}\right) = j - \xi \tag{35}$$

The derivative is

$$dx = -\,h\, d\xi \tag{36}$$

$$d\xi = -\frac{1}{h}dx \tag{37}$$

Note that h is the segment length.

The displacement function becomes.

$$U(\xi) = L_1\, u_{j-1} + L_2\, u_j \ , \qquad 0 \ \le \xi \le 1 \tag{38}$$

The slope equation is

$$U'(\xi) = L_1' u_{j-1} + L_2' u_j \ , \qquad 0 \le \xi \le 1 \tag{39}$$

$$L_1 = \xi \tag{40}$$

$$L_1' = 1 \tag{41}$$

$$L_2 = 1-\xi \tag{42}$$

$$L_2' = -1 \tag{43}$$

Now Let

$$Y(x) = \underline{L}^T \ \bar{a} \ , \qquad (j-1)h \le x \le jh, \quad \xi = j - x/h \tag{44}$$

where

$$\underline{L} = [\, \xi \qquad 1-\xi \,]^T \tag{45}$$

$$\bar{a} = [\, u_{j-1} \qquad u_j \,]^T \tag{46}$$

The derivative terms are

$$\frac{d}{dx} U(x) = \frac{d}{dx} \underline{L}^T \ \bar{a} \ , \qquad (j-1)h \le x \le jh, \quad \xi = j - x/h \tag{47}$$

$$\frac{d}{dx} U(x) = \frac{d}{d\xi} \frac{d\xi}{dx} \underline{L}^T \ \bar{a} \ , \qquad (j-1)h \le x \le jh, \quad \xi = j - x/h \tag{48}$$

$$\frac{d}{dx}U(x) = \left(-\frac{1}{h}\right)\underline{L'}^{T} \quad \bar{a}\,, \qquad (j-1)h \le x \le jh, \quad \xi = j - x/h \tag{49}$$

where

$$\underline{L'} = \begin{bmatrix}1 & -1\end{bmatrix}^{T} \tag{50}$$

Note that primes indicate derivatives with respect to ξ. Recall

$$-EA_0 \int \left\{\left[1+\alpha\frac{x}{L}\right]\left[\frac{d}{dx}\phi(x)\right]\left[\frac{d}{dx}U(x)\right]\right\}dx$$

$$+ \quad \rho A_0 \omega^2 \int \left\{\left[1+\alpha\frac{x}{L}\right]\phi(x)U(x)\right\}dx = 0 \tag{51}$$

The essence of the Galerkin method is that the test function is chosen as

$$\phi(x) = U(x) \tag{52}$$

Thus

$$-EA_0 \int \left\{\left[1+\alpha\frac{x}{L}\right]\left[\frac{d}{dx}U(x)\right]\left[\frac{d}{dx}U(x)\right]\right\}dx$$

$$+ \quad \rho A_0 \omega^2 \int \left\{\left[1+\alpha\frac{x}{L}\right] U(x)U(x)\right\}dx = 0 \tag{53}$$

Change the integration variable using equation. Also, apply the integration limits.

$$- EA_0 h \int_0^1 \left\{ \left[1 + \alpha \frac{(h\,j - h\,\xi)}{L} \right] \left[\frac{d}{dx} U(x) \right] \left[\frac{d}{dx} U(x) \right] \right\} d\xi$$

$$+ \rho A_0 \omega^2 h \int_0^1 \left\{ \left[1 + \alpha \frac{(h\,j - h\,\xi)}{L} \right] U(x) U(x) \right\} d\xi = 0$$

(54)

$$- \frac{EA_0}{L} \int_0^1 \left\{ \left[(L + \alpha h\,j) - \alpha h\,\xi \right] \left[\frac{d}{dx} U(x) \right] \left[\frac{d}{dx} U(x) \right] \right\} d\xi$$

$$+ \frac{\rho A_0 \omega^2}{L} \int_0^1 \left\{ \left[(L + \alpha h\,j) - \alpha h\,\xi \right] U(x) U(x) \right\} d\xi = 0$$

(55)

$$- \frac{EA_0}{L} \int_0^1 \left\{ \left[(L + \alpha h\,j) - \alpha h\,\xi \right] \left[\left(-\frac{1}{h} \right) \underline{L'^{\,T}}\, \bar{a} \right] \left[\left(-\frac{1}{h} \right) \underline{L'^{\,T}}\, \bar{a} \right] \right\} d\xi$$

$$+ \frac{\rho A_0 \omega^2}{L} \int_0^1 \left\{ \left[(L + \alpha h\,j) - \alpha h\,\xi \right] \left[\underline{L}^T \bar{a} \right] \left[\underline{L}^T \bar{a} \right] \right\} d\xi = 0$$

(56)

$$- \frac{EA_0}{L h^2} \int_0^1 \left\{ \left[(L + \alpha h\,j) - \alpha h\,\xi \right] \left[\bar{a}^T \underline{L'}\; \underline{L'}^T \bar{a} \right] \right\} d\xi$$

$$+ \frac{\rho A_0 \omega^2}{L} \int_0^1 \left\{ \left[(L + \alpha h\,j) - \alpha h\,\xi \right] \left[\bar{a}^T \underline{L}\; \underline{L}^T \bar{a} \right] \right\} d\xi = 0$$

(57)

$$
-\frac{EA_0}{L\,h}\int_0^1\left\{\left[\left(L+\alpha h\, j\right)-\alpha h\,\xi\right]\left[\bar{a}^{\,T}\,\underline{L'}\;\underline{L'}^{\,T}\,\bar{a}\right]\right\}d\xi
$$

$$
+\frac{\rho A_0\,h\,\omega^2}{L}\int_0^1\left\{\left[\left(L+\alpha h\, j\right)-\alpha h\,\xi\right]\left[\bar{a}^{\,T}\,\underline{L}\;\underline{L}^{\,T}\,\bar{a}\right]\right\}d\xi=0
$$

(58)

$$
\bar{a}^T\left\{-\frac{EA_0}{Lh}\int_0^1\left\{\left[\left(L+\alpha h\, j\right)-\alpha h\,\xi\right]\left[\underline{L'}\;\underline{L'}^{\,T}\right]\right\}d\xi\right\}\bar{a}
$$

$$
+\,\bar{a}^T\left\{+\frac{\rho A_0\,h\omega^2}{L}\int_0^1\left\{\left[\left(L+\alpha h\, j\right)-\alpha h\,\xi\right]\left[\underline{L}\;\underline{L}^{\,T}\right]\right\}d\xi\right\}\bar{a}\;=0
$$

(59)

$$
-\frac{EA_0}{Lh}\int_0^1\left\{\left[\left(L+\alpha h\, j\right)-\alpha h\,\xi\right]\left[\underline{L'}\;\underline{L'}^{\,T}\right]\right\}d\xi
$$

$$
+\frac{\rho A_0\,h\,\omega^2}{L}\int_0^1\left\{\left[\left(L+\alpha h\, j\right)-\alpha h\,\xi\right]\left[\underline{L}\;\underline{L}^{\,T}\right]\right\}d\xi\;=0
$$

(60)

$$- \frac{E A_0}{Lh} \int_0^1 \left\{ \left[L + \alpha h\, j \right] \left[\underline{L'}\ \ \underline{L'}^T \right] \right\} d\xi$$

$$+ \frac{E A_0}{Lh} \int_0^1 \left\{ \left[\alpha h\, \xi \right] \left[\underline{L'}\ \ \underline{L'}^T \right] \right\} d\xi$$

$$+ \frac{\rho\, A_0\, h\, \omega^2}{L} \int_0^1 \left\{ \left[L + \alpha h\, j \right] \left[\underline{L}\ \ \underline{L}^T \right] \right\} d\xi$$

$$- \frac{\rho\, A_0\, h \omega^2}{L} \int_0^1 \left\{ \left[\alpha h\, \xi \right] \left[\underline{L}\ \ \underline{L}^T \right] \right\} d\xi \quad = \ 0 \tag{61}$$

For a system of n elements,

$$K_j = \omega^2\, M_j \ , \qquad j = 1, 2, ..., n \tag{62}$$

where

$$Kj = \quad + \frac{E A_0}{Lh} \int_0^1 \left\{ \left[L + \alpha h\, j \right] \left[\underline{L'}\ \ \underline{L'}^T \right] \right\} d\xi$$

$$- \frac{\alpha\, E A_0}{L} \int_0^1 \left\{ \xi \ \left[\underline{L'}\ \ \underline{L'}^T \right] \right\} d\xi \tag{63}$$

$$M_j = \ + \ \frac{\rho\, A_0\, h}{L} \int_0^1 \left\{ \left[L + \alpha h\, j \right] \left[\underline{L}\ \ \underline{L}^T \right] \right\} d\xi$$

$$- \frac{\alpha \rho\, A_0\, h^2}{L} \int_0^1 \left\{ \xi \left[\underline{L}\ \ \underline{L}^T \right] \right\} d\xi \tag{64}$$

The first stiffness matrix per Reference 1 is

$$+\frac{EA_0}{Lh}\left[L+\alpha h j\right]\begin{bmatrix}1 & -1\\ & 1\end{bmatrix} \tag{65}$$

The second stiffness matrix per Appendix A.

$$-\frac{\alpha EA_0}{2L}\begin{bmatrix}1 & -1\\ & 1\end{bmatrix} \tag{66}$$

The combined stiffness matrix is

$$K_j=\frac{EA_0}{Lh}\left\{\left[L+\alpha h j\right]\begin{bmatrix}1 & -1\\ & 1\end{bmatrix}-\frac{1}{2}\alpha h\begin{bmatrix}1 & -1\\ & 1\end{bmatrix}\right\} \tag{67}$$

$$K_j=\frac{EA_0}{h}\left\{\left[1+j\alpha\left(\frac{h}{L}\right)\right]\begin{bmatrix}1 & -1\\ & 1\end{bmatrix}-\frac{1}{2}\alpha\left(\frac{h}{L}\right)\begin{bmatrix}1 & -1\\ & 1\end{bmatrix}\right\} \tag{68}$$

$$K_j=\frac{EA_0}{h}\left\{1+j\alpha\left(\frac{h}{L}\right)-\frac{1}{2}\alpha\left(\frac{h}{L}\right)\right\}\begin{bmatrix}1 & -1\\ & 1\end{bmatrix} \tag{69}$$

$$K_j=\frac{EA_0}{h}\left\{1+\frac{\alpha}{2}\left(\frac{h}{L}\right)(2j-1)\right\}\begin{bmatrix}1 & -1\\ & 1\end{bmatrix} \tag{70}$$

The first mass matrix is

$$+\ \frac{\rho\ A_0\,h}{6L}\left[L+\alpha h\,j\right]\begin{bmatrix} 2 & 1 \\ & 2 \end{bmatrix} \tag{71}$$

The second mass matrix is

$$-\ \frac{\alpha\rho\ A_0\,h^2}{12L^2}\begin{bmatrix} 1 & 1 \\ & 3 \end{bmatrix} \tag{72}$$

The combined mass matrix is

$$M_{\,j} = \ +\ \frac{\rho\ A_0\,h}{6L}\left[L+\alpha h\,j\right]\begin{bmatrix} 2 & 1 \\ & 2 \end{bmatrix} - \frac{\alpha\rho\ A_0\,h^2}{12L}\begin{bmatrix} 1 & 1 \\ & 3 \end{bmatrix} \tag{73}$$

$$M_{\,j} = \ +\ \frac{\rho\ A_0\,h}{12L}\left\{2\left[L+\alpha h\,j\right]\begin{bmatrix} 2 & 1 \\ & 2 \end{bmatrix} - \ \alpha h\begin{bmatrix} 1 & 1 \\ & 3 \end{bmatrix}\right\} \tag{74}$$

$$M_{\,j} = \ +\ \frac{\rho\ A_0\,h}{12}\left\{2\left[1+\ j\alpha\left(\frac{h}{L}\right)\right]\begin{bmatrix} 2 & 1 \\ & 2 \end{bmatrix} - \ \alpha\left(\frac{h}{L}\right)\begin{bmatrix} 1 & 1 \\ & 3 \end{bmatrix}\right\} \tag{75}$$

$$M_j = + \frac{\rho A_0 h}{12}\left\{2\begin{bmatrix}2 & 1\\ & 2\end{bmatrix} + 2j\alpha\left(\frac{h}{L}\right)\begin{bmatrix}2 & 1\\ & 2\end{bmatrix} - \alpha\left(\frac{h}{L}\right)\begin{bmatrix}1 & 1\\ & 3\end{bmatrix}\right\} \tag{76}$$

$$M_j = + \frac{\rho A_0 h}{12}\left\{\begin{bmatrix}4 & 2\\ & 4\end{bmatrix} + \alpha\left(\frac{h}{L}\right)\left[j\begin{bmatrix}4 & 2\\ & 4\end{bmatrix} - \begin{bmatrix}1 & 1\\ & 3\end{bmatrix}\right]\right\} \tag{77}$$

$$M_j = + \frac{\rho A_0 h}{12}\left\{\begin{bmatrix}4 & 2\\ & 4\end{bmatrix} + \alpha\left(\frac{h}{L}\right)\begin{bmatrix}4j-1 & 2j-1\\ & 4j-3\end{bmatrix}\right\} \tag{78}$$

References

1. T. Irvine, Longitudinal Vibration of a Rod via the Finite Element Method, Vibrationdata, 2003.

2. T. Irvine, Longitudinal Vibration of Tapered Rod, Vibrationdata, 2003.

APPENDIX A

$$\int_0^1 \left\{ \xi \left[\underline{L'}\ \underline{L'}^T \right] \right\} d\xi$$

$$= \int_0^1 \left\{ \xi \begin{bmatrix} 1 \\ \\ -1 \end{bmatrix} \begin{bmatrix} 1 & & -1 \end{bmatrix} \right\} d\xi$$

$$= \int_0^1 \left\{ \xi \begin{bmatrix} 1 & -1 \\ & 1 \end{bmatrix} \right\} d\xi$$

$$= \begin{bmatrix} 1 & -1 \\ & 1 \end{bmatrix} \int_0^1 \xi \, d\xi$$

$$= \begin{bmatrix} 1 & -1 \\ & 1 \end{bmatrix} \frac{1}{2} \xi^2 \Big|_0^1$$

$$= \frac{1}{2} \begin{bmatrix} 1 & -1 \\ & 1 \end{bmatrix}$$

APPENDIX B

$$\int_0^1 \left\{ \xi \left[\underline{L} \ \underline{L}^T \right] \right\} d\xi$$

$$= \int_0^1 \left\{ \xi \begin{bmatrix} \xi \\ \\ 1-\xi \end{bmatrix} \begin{bmatrix} \xi & 1-\xi \end{bmatrix} \right\} d\xi$$

$$= \int_0^1 \left\{ \xi \begin{bmatrix} \xi^2 & (\xi)(1-\xi) \\ \\ & (1-\xi)(1-\xi) \end{bmatrix} \right\} d\xi$$

$$= \int_0^1 \left\{ \begin{bmatrix} \xi^3 & \left(\xi^2 - \xi^3\right) \\ \\ & \left(\xi - 2\xi^2 + \xi^3\right) \end{bmatrix} \right\} d\xi$$

$$= \left. \begin{bmatrix} \frac{1}{4}\xi^4 & \left(\frac{1}{3}\xi^3 - \frac{1}{4}\xi^4\right) \\ \\ & \left(\frac{1}{2}\xi^2 - \frac{2}{3}\xi^3 + \frac{1}{4}\xi^4\right) \end{bmatrix} \right|_0^1$$

$$= \begin{bmatrix} \frac{1}{4} & \frac{1}{12} \\ & \frac{1}{12} \end{bmatrix}$$

$$= \frac{1}{12}\begin{bmatrix} 3 & 1 \\ & 1 \end{bmatrix}$$

APPENDIX C

Fixed-Free Rod with $A_L = 0$

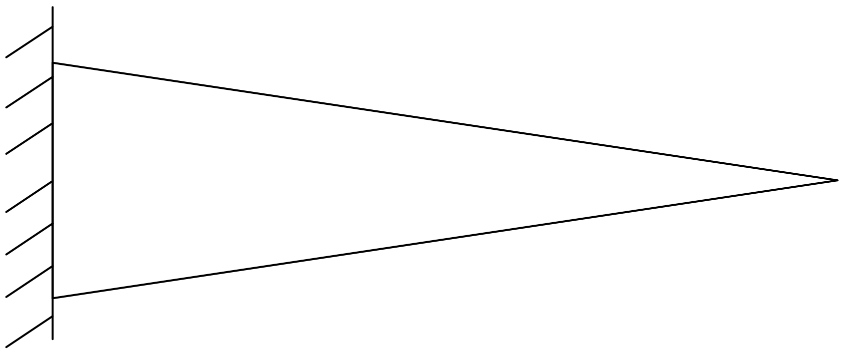


Figure C-1.

$$\alpha = \frac{A_L}{A_0} - 1 \tag{C-1a}$$

$$A_L = 0 \tag{C-1b}$$

$$\alpha = -1 \tag{C-1c}$$

Model the rod with two elements.

$$h = L/2 \tag{C-2}$$

Furthermore, the rod is fixed at x=0 and free at x=L.

$$U(0) = 0 \tag{C-3}$$

$$U'(L) = 0 \tag{C-4}$$

$$K_j = \frac{2EA_0}{L}\left\{1 + \frac{-1}{4}\left(2j-1\right)\right\}\begin{bmatrix}1 & -1\\ & 1\end{bmatrix} \tag{C-5}$$

$$K_1 = \frac{2EA_0}{L}\left\{1 + \frac{-1}{4}\right\}\begin{bmatrix}1 & -1\\ & 1\end{bmatrix} \tag{C-6}$$

$$K_1 = \frac{2EA_0}{L}\left\{\frac{3}{4}\right\}\begin{bmatrix}1 & -1\\ & 1\end{bmatrix} \tag{C-7}$$

$$K_1 = \frac{3EA_0}{2L}\begin{bmatrix}1 & -1\\ & 1\end{bmatrix} \tag{C-8}$$

$$K_1 = \frac{EA_0}{L}\begin{bmatrix}1.5 & -1.5\\ & 1.5\end{bmatrix} \tag{C-9}$$

$$K_2 = \frac{2EA_0}{L}\left\{1 + \frac{-3}{4}\right\}\begin{bmatrix}1 & -1\\ & 1\end{bmatrix} \tag{C-10}$$

$$K_2 = \frac{2EA_0}{L}\left\{\frac{1}{4}\right\}\begin{bmatrix}1 & -1\\ & 1\end{bmatrix} \tag{C-11}$$

$$K_2 = \frac{EA_0}{2L}\begin{bmatrix}1 & -1\\ & 1\end{bmatrix} \tag{C-11}$$

$$K_2 = \frac{E\,A_0}{L}\begin{bmatrix} 0.5 & -0.5 \\ & 0.5 \end{bmatrix} \tag{C-12}$$

$$K_1 = \frac{E\,A_0}{L}\begin{bmatrix} 1.5 & -1.5 \\ & 1.5 \end{bmatrix} \tag{C-13}$$

$$K = \frac{E\,A_0}{L}\begin{bmatrix} 1.5 & -1.5 & 0 \\ & 2 & -0.5 \\ & & 0.5 \end{bmatrix} \tag{C-14}$$

$$M_j = \quad + \ \frac{\rho\,A_0\,h}{12}\left\{2\left[1 - \frac{j}{2}\right]\begin{bmatrix} 2 & 1 \\ & 2 \end{bmatrix} - \ \alpha\left(\frac{h}{L}\right)\begin{bmatrix} 3 & 1 \\ & 1 \end{bmatrix}\right\} \tag{C-15}$$

$$M_j = \quad + \ \frac{\rho\,A_0\,L}{24}\left\{2\left[1 - \left(\frac{j}{2}\right)\right]\begin{bmatrix} 2 & 1 \\ & 2 \end{bmatrix} + \left(\frac{1}{2}\right)\begin{bmatrix} 3 & 1 \\ & 1 \end{bmatrix}\right\} \tag{C-16}$$

$$M_1 = \quad + \ \frac{\rho\,A_0\,L}{24}\left\{2\left[1 - \left(\frac{1}{2}\right)\right]\begin{bmatrix} 2 & 1 \\ & 2 \end{bmatrix} + \left(\frac{1}{2}\right)\begin{bmatrix} 3 & 1 \\ & 1 \end{bmatrix}\right\} \tag{C-17}$$

$$M_1 = \quad + \ \frac{\rho\,A_0\,L}{24}\left\{2\left[\frac{1}{2}\right]\begin{bmatrix} 2 & 1 \\ & 2 \end{bmatrix} + \left(\frac{1}{2}\right)\begin{bmatrix} 3 & 1 \\ & 1 \end{bmatrix}\right\} \tag{C-18}$$

$$M_1 = \; + \; \frac{\rho\, A_0\, L}{48}\left\{ 2\begin{bmatrix} 2 & 1 \\ & 2 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ & 1 \end{bmatrix}\right\} \tag{C-19}$$

$$M_1 = \; + \; \frac{\rho\, A_0\, L}{48}\left\{ \begin{bmatrix} 4 & 2 \\ & 4 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ & 1 \end{bmatrix}\right\} \tag{C-20}$$

$$M_1 = \; + \; \frac{\rho\, A_0\, L}{48}\begin{bmatrix} 7 & 3 \\ & 5 \end{bmatrix} \tag{C-21}$$

$$M_2 = \; + \; \frac{\rho\, A_0\, L}{24}\left\{ 2\left[1 - \left(\frac{2}{2}\right) \right]\begin{bmatrix} 2 & 1 \\ & 2 \end{bmatrix} + \left(\frac{1}{2}\right)\begin{bmatrix} 3 & 1 \\ & 1 \end{bmatrix}\right\} \tag{C-22}$$

$$M_2 = \; + \; \frac{\rho\, A_0\, L}{24}\left\{ 2\,[1 - 1]\begin{bmatrix} 2 & 1 \\ & 2 \end{bmatrix} + \left(\frac{1}{2}\right)\begin{bmatrix} 3 & 1 \\ & 1 \end{bmatrix}\right\} \tag{C-23}$$

$$M_2 = \; + \; \frac{\rho\, A_0\, L}{24}\left\{ \left(\frac{1}{2}\right)\begin{bmatrix} 3 & 1 \\ & 1 \end{bmatrix}\right\} \tag{C-24}$$

$$M_2 = \; + \; \frac{\rho\, A_0\, L}{48}\begin{bmatrix} 3 & 1 \\ & 1 \end{bmatrix} \tag{C-25}$$

$$M_1 = \; + \; \frac{\rho\, A_0\, L}{48}\begin{bmatrix} 7 & 3 \\ & 5 \end{bmatrix} \tag{C-26}$$

$$M \; = \; + \; \frac{\rho\, A_0\, L}{48}\begin{bmatrix} 7 & 3 & 0 \\ & 8 & 1 \\ & & 1 \end{bmatrix} \tag{C-26}$$

The global assembly for the model with two elements is
0

$$\det\left\{\frac{E A_0}{L}\begin{bmatrix}1.5 & -1.5 & 0\\ & 2 & -0.5\\ & & 0.5\end{bmatrix}-\frac{\rho\, A_0\, L\, \omega^2}{48}\begin{bmatrix}7 & 3 & 0\\ & 8 & 1\\ & & 1\end{bmatrix}\right\} = 0$$

(C-27)

$$\det\left\{\frac{E}{L}\begin{bmatrix}1.5 & -1.5 & 0\\ & 2 & -0.5\\ & & 0.5\end{bmatrix}-\frac{\rho\, L\, \omega^2}{48}\begin{bmatrix}7 & 3 & 0\\ & 8 & 1\\ & & 1\end{bmatrix}\right\} = 0$$

(C-28)

$$\det\left\{\begin{bmatrix}1.5 & -1.5 & 0\\ & 2 & -0.5\\ & & 0.5\end{bmatrix}-\lambda\begin{bmatrix}7 & 3 & 0\\ & 8 & 1\\ & & 1\end{bmatrix}\right\} = 0$$

(C-29)

$$\lambda = \frac{\rho\, L^2\, \omega^2}{48 E} \qquad \text{(C-30)}$$

Apply the boundary conditions for the fixed-free case. The first row and column are removed for this purpose.

$$\det\left\{\begin{bmatrix}2 & -0.5\\ & 0.5\end{bmatrix}-\lambda\begin{bmatrix}8 & 1\\ & 1\end{bmatrix}\right\} = 0 \qquad \text{(C-31)}$$

$$\lambda_1 = 0.122 \tag{C-32}$$

$$\lambda_2 = 0.878 \tag{C-33}$$

$$\frac{\rho L^2 \omega_1{}^2}{48 E} = 0.122 \tag{C-34}$$

$$\omega_1 = \frac{2.42}{L}\sqrt{\frac{E}{\rho}} \qquad \text{(FE analysis)} \tag{C-35}$$

The theoretical value from Reference 2 is

$$\omega_1 = \frac{2.405}{L}\sqrt{\frac{E}{\rho}} \qquad \text{(theoretical)} \tag{C-36}$$

Thus, the agreement is excellent, within 1%.