RESPONSE OF A SINGLE-DEGREE-OF-FREEDOM SYSTEM SUBJECTED TO A TERMINAL SAWTOOTH PULSE BASE EXCITATION

Revision B

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Introduction

Consider the single-degree-of-freedom system in Figure 1.

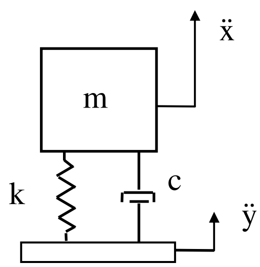

Figure 1.

where

m is the mass
c is the viscous damping coefficient
k is the stiffness
x is the absolute displacement of the mass
y is the base input displacement

A free-body diagram is shown in Figure 2.

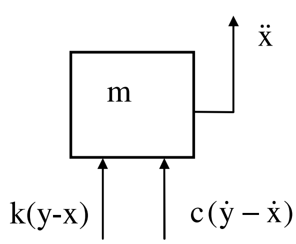

Figure 2.

Summation of forces in the vertical direction

$$\sum F = m\ddot{x} \tag{1}$$

$$m\ddot{x} = c\,(\dot{y} - \dot{x}) + k\,(y - x) \tag{2}$$

$$\begin{aligned} \text{Let } z &= x - y \quad \text{(relative displacement)} \\ \dot{z} &= \dot{x} - \dot{y} \\ \ddot{z} &= \ddot{x} - \ddot{y} \\ \ddot{x} &= \ddot{z} + \ddot{y} \end{aligned}$$

Substituting the relative displacement terms into equation (2) yields

$$m(\ddot{z} + \ddot{y}) = -c\dot{z} - kz \tag{3}$$

$$m\ddot{z} + c\dot{z} + kz = -m\ddot{y} \tag{4}$$

Dividing through by mass yields

$$\ddot{z} + (c\,/\,m)\dot{z} + (k\,/\,m)z = -\ddot{y} \tag{5}$$

By convention,

$$\begin{aligned} (c\,/\,m) &= 2\xi\omega_n \\ (k\,/\,m) &= \omega_n{}^2 \end{aligned}$$

where ω_n is the natural frequency in (radians/sec), and ξ is the damping ratio.

Substitute the convention terms into equation (5).

$$\ddot{z} + 2\xi\omega_n\dot{z} + \omega_n{}^2 z = -\ddot{y} \tag{6}$$

Terminal Sawtooth Pulse

Consider the pulse given by equation (7).

$$\ddot{y}(t) = \begin{cases} A\left(\dfrac{t}{T}\right), & 0 \le t \le T \\ \\ 0, & t > T \end{cases} \tag{7}$$

The equation of motion becomes

$$\ddot{z} + 2\xi\omega_n\dot{z} + \omega_n{}^2 z = -A\left(\frac{t}{T}\right), \qquad 0 \le t \le T \tag{8}$$

Now take the Laplace transform.

$$L\left\{\ddot{z} + 2\xi\omega_n\dot{z} + \omega_n^2 z\right\} = L\{-At/T\} \tag{9}$$

$$\begin{aligned} &s^2 Z(s) - s z(0) - \dot{z}(0) \\ &+ 2\xi\omega_n s Z(s) - 2\xi\omega_n z(0) \\ &+ \omega_n{}^2 Z(s) \quad = \frac{-A/T}{s^2} \end{aligned} \tag{10}$$

$$\left\{s^2 + 2\xi\omega_n s + \omega_n{}^2\right\}Z(s) + \{-1\}\dot{z}(0) + \{-s - 2\xi\omega_n\}z(0) = \frac{-A/T}{s^2} \tag{11}$$

$$\left\{s^2 + 2\xi\omega_n s + \omega_n{}^2\right\}Z(s) = \dot{z}(0) + \{s + 2\xi\omega_n\}z(0) - \frac{A/T}{s^2} \tag{12}$$

$$Z(s) = \left\{ \frac{\dot{z}(0) + \{s + 2\xi\omega_n\} z(0)}{s^2 + 2\xi\omega_n s + \omega_n^2} \right\} - \left\{ \frac{A/T}{s^2} \right\} \left\{ \frac{1}{s^2 + 2\xi\omega_n s + \omega_n^2} \right\} \tag{13}$$

Let

$$Z(s) = Z_n(s) + Z_f(s) \tag{14}$$

where

$$Z_n(s) = \left\{ \frac{\dot{z}(0) + \{s + 2\xi\omega_n\} z(0)}{s^2 + 2\xi\omega_n s + \omega_n^2} \right\} \tag{15}$$

$$Z_f(s) = -\left\{ \frac{A/T}{s^2} \right\} \left\{ \frac{1}{s^2 + 2\xi\omega_n s + \omega_n^2} \right\} \tag{16}$$

Consider the denominator term,

$$s^2 + 2\xi\omega_n s + \omega_n^2 = (s + \xi\omega_n)^2 + \omega_n^2 - (\xi\omega_n)^2 \tag{17}$$

$$s^2 + 2\xi\omega_n s + \omega_n^2 = (s + \xi\omega_n)^2 + \omega_n^2 (1 - \xi^2) \tag{18}$$

Now define the damped natural frequency,

$$\omega_d = \omega_n \sqrt{1 - \xi^2} \tag{19}$$

Substitute equation (19) into (18),

$$s^2 + 2\xi\omega_n s + \omega_n^2 = (s + \xi\omega_n)^2 + \omega_d^2 \tag{20}$$

Substitute equation (20) into (16).

$$Z_n(s) = \left\{ \frac{\dot{z}(0) + \{s + 2\xi\omega_n\} z(0)}{(s + \xi\omega_n)^2 + \omega_d^2} \right\} \tag{21}$$

Rearrange the terms into a convenient format prior to the inverse Laplace transform.

$$Z_n(s) = \left\{ \frac{(s+\xi\omega_n)z(0)}{(s+\xi\omega_n)^2+\omega_d{}^2} \right\} + \left\{ \frac{\dot{z}(0)+(\xi\omega_n)z(0)}{(s+\xi\omega_n)^2+\omega_d{}^2} \right\} \tag{22}$$

$$Z_n(s) = \left\{ \frac{(s+\xi\omega_n)z(0)}{(s+\xi\omega_n)^2+\omega_d{}^2} \right\} + \left\{ \frac{\left\{ \frac{\dot{z}(0)+(\xi\omega_n)z(0)}{\omega_d} \right\}\omega_d}{(s+\xi\omega_n)^2+\omega_d{}^2} \right\} \tag{23}$$

Take the inverse Laplace transform using Reference 1.

$$z_n(t) = z(0)\exp(-\xi\omega_n t)\cos(\omega_d t) + \left\{ \frac{\dot{z}(0)+(\xi\omega_n)z(0)}{\omega_d} \right\}\exp(-\xi\omega_n t)\sin(\omega_d t) \tag{24}$$

$$z_n(t) = \exp(-\xi\omega_n t)\left\{ z(0)\cos(\omega_d t) + \left\{ \frac{\dot{z}(0)+(\xi\omega_n)z(0)}{\omega_d} \right\}\sin(\omega_d t) \right\} \tag{25}$$

Take the first derivative to determine the relative velocity.

$$\begin{aligned} \dot{z}_n(t) = & -\xi\omega_n \exp(-\xi\omega_n t)\left\{ z(0)\cos(\omega_d t) + \left\{ \frac{\dot{z}(0)+(\xi\omega_n)z(0)}{\omega_d} \right\}\sin(\omega_d t) \right\} \\ & + \exp(-\xi\omega_n t)\{-\omega_d z(0)\sin(\omega_d t) + \{\dot{z}(0)+(\xi\omega_n)z(0)\}\cos(\omega_d t)\} \end{aligned} \tag{26}$$

$$\begin{aligned} \dot{z}_n(t) = & \exp(-\xi\omega_n t)\left\{ -\xi\omega_n z(0)\cos(\omega_d t) - \xi\omega_n \left\{ \frac{\dot{z}(0)+(\xi\omega_n)z(0)}{\omega_d} \right\}\sin(\omega_d t) \right\} \\ & + \exp(-\xi\omega_n t)\{-\omega_d z(0)\sin(\omega_d t) + \{\dot{z}(0)+(\xi\omega_n)z(0)\}\cos(\omega_d t)\} \end{aligned} \tag{27}$$

$$\dot{z}_n(t) =$$

$$\exp(-\xi\omega_n t)\{\{-\xi\omega_n z(0) + \dot{z}(0) + (\xi\omega_n)z(0)\}\cos(\omega_d t)\}$$

$$+\exp(-\xi\omega_n t)\left\{\left\{-\omega_d z(0) - \xi\omega_n\left[\frac{\dot{z}(0) + (\xi\omega_n)z(0)}{\omega_d}\right]\right\}\sin(\omega_d t)\right\}$$

(28)

$$\dot{z}_n(t) = \exp(-\xi\omega_n t)\{\dot{z}(0)\cos(\omega_d t)\}$$

$$+\exp(-\xi\omega_n t)\left\{\frac{1}{\omega_d}\left\{-\omega_d{}^2 z(0) - \xi\omega_n[\dot{z}(0) + (\xi\omega_n)z(0)]\right\}\sin(\omega_d t)\right\}$$

(29)

$$\dot{z}_n(t) = \exp(-\xi\omega_n t)\{\dot{z}(0)\cos(\omega_d t)\}$$

$$+\exp(-\xi\omega_n t)\left\{\frac{1}{\omega_d}\left\{-\omega_d{}^2 z(0) - \xi\omega_n\dot{z}(0) - (\xi\omega_n)^2 z(0)\right\}\sin(\omega_d t)\right\}$$

(30)

$$\dot{z}_n(t) = \exp(-\xi\omega_n t)\{\dot{z}(0)\cos(\omega_d t)\}$$

$$+\exp(-\xi\omega_n t)\left\{\frac{1}{\omega_d}\left\{-\omega_n{}^2\left(1-\xi^2\right)z(0) - \xi\omega_n\dot{z}(0) - (\xi\omega_n)^2 z(0)\right\}\sin(\omega_d t)\right\}$$

(31)

$$\dot{z}_n(t) = \exp(-\xi\omega_n t)\{\dot{z}(0)\cos(\omega_d t)\}$$

$$+\exp(-\xi\omega_n t)\left\{\frac{1}{\omega_d}\left\{\left(-\omega_n{}^2 + \xi^2\omega_n{}^2\right)z(0) - \xi\omega_n\dot{z}(0) - (\xi\omega_n)^2 z(0)\right\}\sin(\omega_d t)\right\}$$

(32)

$$\dot{z}_n(t) = \exp(-\xi\omega_n t)\{\dot{z}(0)\cos(\omega_d t)\}$$

$$+\exp(-\xi\omega_n t)\left\{\frac{1}{\omega_d}\left\{-\omega_n{}^2 z(0) - \xi\omega_n\dot{z}(0)\right\}\sin(\omega_d t)\right\} \tag{33}$$

$$\dot{z}_n(t) = \exp(-\xi\omega_n t)\left\{\dot{z}(0)\cos(\omega_d t) + \frac{1}{\omega_d}\left\{-\omega_n{}^2 z(0) - \xi\omega_n\dot{z}(0)\right\}\sin(\omega_d t)\right\} \tag{34}$$

$$\dot{z}_n(t) = \exp(-\xi\omega_n t)\left\{\dot{z}(0)\cos(\omega_d t) + \frac{\omega_n}{\omega_d}\{-\omega_n z(0) - \xi\dot{z}(0)\}\sin(\omega_d t)\right\} \tag{35}$$

Take the second derivative to determine the acceleration.

$$\ddot{z}_n(t) = -\xi\omega_n\exp(-\xi\omega_n t)\left\{\dot{z}(0)\cos(\omega_d t) + \frac{\omega_n}{\omega_d}\{-\omega_n z(0) - \xi\dot{z}(0)\}\sin(\omega_d t)\right\}$$
$$+\exp(-\xi\omega_n t)\{-\omega_d\dot{z}(0)\sin(\omega_d t) + \omega_n\{-\omega_n z(0) - \xi\dot{z}(0)\}\cos(\omega_d t)\} \tag{36}$$

$$\ddot{z}_n(t) = \exp(-\xi\omega_n t)\left\{-\xi\omega_n\dot{z}(0)\cos(\omega_d t) - \frac{\xi\omega_n{}^2}{\omega_d}\{-\omega_n z(0) - \xi\dot{z}(0)\}\sin(\omega_d t)\right\}$$
$$+\exp(-\xi\omega_n t)\{-\omega_d\dot{z}(0)\sin(\omega_d t) + \omega_n\{-\omega_n z(0) - \xi\dot{z}(0)\}\cos(\omega_d t)\} \tag{37}$$

$$\ddot{z}_n(t) = \exp(-\xi\omega_n t)\{-\xi\omega_n \dot{z}(0) + \omega_n\{-\omega_n z(0) - \xi \dot{z}(0)\}\}\cos(\omega_d t)$$
$$+\exp(-\xi\omega_n t)\left\{-\omega_d \dot{z}(0) - \frac{\xi\omega_n{}^2}{\omega_d}\{-\omega_n z(0) - \xi \dot{z}(0)\}\right\}\sin(\omega_d t) \tag{38}$$

$$\ddot{z}_n(t) = -\omega_n \exp(-\xi\omega_n t)\{\omega_n z(0) + 2\xi \dot{z}(0)\}\cos(\omega_d t)$$
$$+\frac{1}{\omega_d}\exp(-\xi\omega_n t)\left\{-\omega_d{}^2 \dot{z}(0) - \xi\omega_n{}^2\{-\omega_n z(0) - \xi \dot{z}(0)\}\right\}\sin(\omega_d t) \tag{39}$$

$$\ddot{z}_n(t) = -\omega_n \exp(-\xi\omega_n t)\{\omega_n z(0) + 2\xi \dot{z}(0)\}\cos(\omega_d t)$$
$$+\frac{1}{\omega_d}\exp(-\xi\omega_n t)\left\{-\omega_d{}^2 \dot{z}(0) + \xi\omega_n{}^3 z(0) + \xi^2\omega_n{}^2 \dot{z}(0)\right\}\sin(\omega_d t) \tag{40}$$

$$\ddot{z}_n(t) = -\omega_n \exp(-\xi\omega_n t)\{\omega_n z(0) + 2\xi \dot{z}(0)\}\cos(\omega_d t)$$
$$+\frac{1}{\omega_d}\exp(-\xi\omega_n t)\left\{-\omega_n{}^2\left(1-\xi^2\right)\dot{z}(0) + \xi\omega_n{}^3 z(0) + \xi^2\omega_n{}^2 \dot{z}(0)\right\}\sin(\omega_d t) \tag{41}$$

$$\ddot{z}_n(t) = -\omega_n \exp(-\xi\omega_n t)\{\omega_n z(0) + 2\xi \dot{z}(0)\}\cos(\omega_d t)$$
$$+\frac{\omega_n{}^2}{\omega_d}\exp(-\xi\omega_n t)\left\{-\left(1-\xi^2\right)\dot{z}(0) + \xi\omega_n z(0) + \xi^2 \dot{z}(0)\right\}\sin(\omega_d t) \tag{42}$$

$$\ddot{z}_n(t) = -\omega_n \exp(-\xi\omega_n t)\{\omega_n z(0) + 2\xi \dot{z}(0)\}\cos(\omega_d t)$$
$$+\frac{\omega_n{}^2}{\omega_d}\exp(-\xi\omega_n t)\left\{\xi\omega_n z(0) + \left(-1+2\xi^2\right)\dot{z}(0)\right\}\sin(\omega_d t) \tag{43}$$

$$\ddot{z}_n(t) = -\omega_n \exp(-\xi\omega_n t)\{\omega_n z(0) + 2\xi\,\dot{z}(0)\}\cos(\omega_d t)$$
$$-\frac{\omega_n{}^2}{\omega_d}\exp(-\xi\omega_n t)\left\{-\xi\omega_n z(0) + \left(1-2\xi^2\right)\dot{z}(0)\right\}\sin(\omega_d t)$$

(44)

$$\ddot{z}_n(t) =$$
$$-\exp(-\xi\omega_n t)\left\{\omega_n\left[\omega_n z(0) + 2\xi\,\dot{z}(0)\right]\cos(\omega_d t) + \frac{\omega_n{}^2}{\omega_d}\left[-\xi\omega_n z(0) + \left(1-2\xi^2\right)\dot{z}(0)\right]\sin(\omega_d t)\right\}$$

(45)

$$\ddot{z}_n(t) =$$
$$-\omega_n \exp(-\xi\omega_n t)\left\{\left[\omega_n z(0) + 2\xi\,\dot{z}(0)\right]\cos(\omega_d t) + \frac{\omega_n}{\omega_d}\left[-\xi\omega_n z(0) + \left(1-2\xi^2\right)\dot{z}(0)\right]\sin(\omega_d t)\right\}$$

(46)

Recall equation (16).

$$Z_f(s) = -\left\{\frac{A/T}{s^2}\right\}\left\{\frac{1}{s^2 + 2\xi\omega_n s + \omega_n{}^2}\right\} \quad (47)$$

Expand into partial fractions using Appendix A.

$$\left\{\frac{1}{s^2}\right\}\left\{\frac{1}{s^2+\alpha s+\beta}\right\} = \left\{\frac{1}{\beta^2}\right\}\left\{\frac{-\alpha s+\beta}{s^2}\right\} + \left\{\frac{1}{\beta^2}\right\}\left\{\frac{\alpha s+\alpha^2-\beta}{s^2+\alpha s+\beta}\right\} \quad (48)$$

$$\alpha = 2\xi\omega_n \quad (49)$$

$$\beta = \omega_n{}^2 \quad (50)$$

$$Z_f(s) = -\left\{\frac{A/T}{\omega_n{}^4}\right\}\left\{\frac{-2\xi\omega_n s + \omega_n{}^2}{s^2} + \frac{2\xi\omega_n s + (2\xi\omega_n)^2 - \omega_n{}^2}{s^2 + 2\xi\omega_n s + \omega_n{}^2}\right\} \tag{51}$$

$$Z_f(s) = -\left\{\frac{A/T}{\omega_n{}^3}\right\}\left\{\frac{-2\xi\, s + \omega_n}{s^2} + \frac{2\xi\, s + (2\xi)^2\omega_n - \omega_n}{s^2 + 2\xi\omega_n s + \omega_n{}^2}\right\} \tag{52}$$

$$Z_f(s) = -\left\{\frac{A/T}{\omega_n{}^3}\right\}\left\{\frac{-2\xi}{s} + \frac{\omega_n}{s^2} + \frac{2\xi\, s + (2\xi)^2\omega_n - \omega_n}{s^2 + 2\xi\omega_n s + \omega_n{}^2}\right\} \tag{53}$$

$$s^2 + 2\xi\omega_n s + \omega_n{}^2 = (s + \xi\omega_n)^2 + \omega_d{}^2 \tag{54}$$

$$Z_f(s) = -\left\{\frac{A/T}{\omega_n{}^3}\right\}\left\{\frac{-2\xi}{s} + \frac{\omega_n}{s^2} + \frac{2\xi\, s + (2\xi)^2\omega_n - \omega_n}{(s + \xi\omega_n)^2 + \omega_d{}^2}\right\} \tag{55}$$

$$Z_f(s) = -\left\{\frac{A/T}{\omega_n{}^3}\right\}\left\{\frac{-2\xi}{s} + \frac{\omega_n}{s^2} + \frac{2\xi\, s + \omega_n\left[(2\xi)^2 - 1\right]}{(s + \xi\omega_n)^2 + \omega_d{}^2}\right\} \tag{56}$$

$$Z_f(s) = -\left\{\frac{A/T}{\omega_n{}^3}\right\}\left\{\frac{-2\xi}{s} + \frac{\omega_n}{s^2} + [2\xi]\left[\frac{s + \omega_n\left[(2\xi) - \frac{1}{2\xi}\right]}{(s + \xi\omega_n)^2 + \omega_d{}^2}\right]\right\} \tag{57}$$

$$Z_f(s) = -\left\{\frac{A/T}{\omega_n{}^3}\right\}\left\{\frac{-2\xi}{s} + \frac{\omega_n}{s^2} + [2\xi]\left[\frac{s + \frac{\omega_n}{2\xi}\left[(2\xi)^2 - 1\right]}{(s+\xi\omega_n)^2 + \omega_d{}^2}\right]\right\} \tag{58}$$

$$Z_f(s) = -\left\{\frac{A/T}{\omega_n{}^3}\right\}\left\{\frac{-2\xi}{s} + \frac{\omega_n}{s^2} + [2\xi]\left[\frac{s + \frac{\omega_n}{2\xi}\left[(2\xi)^2 - 1\right]}{(s+\xi\omega_n)^2 + \omega_d{}^2}\right]\right\} \tag{59}$$

$$Z_f(s) = -\left\{\frac{A/T}{\omega_n{}^3}\right\}\left\{\frac{-2\xi}{s} + \frac{\omega_n}{s^2} + 2\xi\left[\frac{s + 2\xi\omega_n}{(s+\xi\omega_n)^2 + \omega_d{}^2}\right] + \left[\frac{-\ \omega_n}{(s+\xi\omega_n)^2 + \omega_d{}^2}\right]\right\} \tag{60}$$

$$Z_f(s) = -\left\{\frac{A/T}{\omega_n{}^3}\right\}\left\{\frac{-2\xi}{s} + \frac{\omega_n}{s^2} + 2\xi\left[\frac{s + 2\xi\omega_n}{(s+\xi\omega_n)^2 + \omega_d{}^2}\right] + \left[\frac{-\ \omega_n}{(s+\xi\omega_n)^2 + \omega_d{}^2}\right]\right\} \tag{61}$$

$$Z_f(s) = -\left\{\frac{A/T}{\omega_n{}^3}\right\}\left\{\frac{-2\xi}{s} + \frac{\omega_n}{s^2} + 2\xi\left[\frac{s + \xi\omega_n}{(s+\xi\omega_n)^2 + \omega_d{}^2}\right] + \left[\frac{2\xi^2\omega_n - \omega_n}{(s+\xi\omega_n)^2 + \omega_d{}^2}\right]\right\} \tag{62}$$

$$Z_f(s) = -\left\{\frac{A/T}{\omega_n{}^3}\right\}\left\{\frac{-2\xi}{s} + \frac{\omega_n}{s^2} + 2\xi\left[\frac{s + \xi\omega_n}{(s+\xi\omega_n)^2 + \omega_d{}^2}\right] + \omega_n\left[\frac{2\xi^2 - 1}{(s+\xi\omega_n)^2 + \omega_d{}^2}\right]\right\} \tag{63}$$

$$Z_f(s) = -\left\{\frac{A/T}{\omega_n{}^2}\right\}\left\{\frac{-2\xi}{\omega_n s} + \frac{1}{s^2} + \left[\frac{2\xi}{\omega_n}\right]\left[\frac{s+\xi\omega_n}{(s+\xi\omega_n)^2 + \omega_d{}^2}\right] + \left[\frac{2\xi^2 - 1}{(s+\xi\omega_n)^2 + \omega_d{}^2}\right]\right\}$$

(64)

$$Z_f(s) = \left\{\frac{A/T}{\omega_n{}^2}\right\}\left\{\frac{2\xi}{\omega_n s} - \frac{1}{s^2} - \left[\frac{2\xi}{\omega_n}\right]\left[\frac{s+\xi\omega_n}{(s+\xi\omega_n)^2 + \omega_d{}^2}\right] - \left[\frac{2\xi^2 - 1}{(s+\xi\omega_n)^2 + \omega_d{}^2}\right]\right\}$$

(65)

Take the inverse Laplace transform using Reference 1. The relative displacement is

$$z_f(t) = \frac{A}{\omega_n{}^2 T}\left\{\frac{2\xi}{\omega_n} - t - \exp(-\xi\omega_n t)\left[\frac{2\xi}{\omega_n}\cos(\omega_d t) + \frac{1}{\omega_d}\left(2\xi^2 - 1\right)\sin(\omega_d t)\right]\right\}$$

(66)

$$z_n(t) = \exp(-\xi\omega_n t)\left\{z(0)\cos(\omega_d t) + \left\{\frac{\dot{z}(0) + (\xi\omega_n)z(0)}{\omega_d}\right\}\sin(\omega_d t)\right\} \tag{67}$$

The total relative displacement is

$$z(t) = \exp(-\xi\omega_n t)\left\{z(0)\cos(\omega_d t) + \left\{\frac{\dot{z}(0) + (\xi\omega_n)z(0)}{\omega_d}\right\}\sin(\omega_d t)\right\}$$

$$+\frac{A}{\omega_n{}^2 T}\left\{\frac{2\xi}{\omega_n} - t - \exp(-\xi\omega_n t)\left[\frac{2\xi}{\omega_n}\cos(\omega_d t) + \frac{1}{\omega_d}\left(2\xi^2 - 1\right)\sin(\omega_d t)\right]\right\},$$

$$0 \le t \le T$$

(68)

The relative velocity is

$$\dot{z}(t) = \exp(-\xi\omega_n t)\left\{\dot{z}(0)\cos(\omega_d t) + \frac{\omega_n}{\omega_d}\{-\omega_n z(0) - \xi\dot{z}(0)\}\sin(\omega_d t)\right\}$$

$$+\frac{A}{\omega_n{}^2 T}\left\{\xi\omega_n \exp(-\xi\omega_n t)\left[\frac{2\xi}{\omega_n}\cos(\omega_d t) + \frac{1}{\omega_d}\left(2\xi^2 - 1\right)\sin(\omega_d t)\right]\right\}$$

$$+\frac{A}{\omega_n{}^2 T}\left\{-\exp(-\xi\omega_n t)\left[\frac{-2\xi\,\omega_d}{\omega_n}\sin(\omega_d t) + \left(2\xi^2 - 1\right)\cos(\omega_d t)\right]\right\}$$

$$0 \le t \le T$$

(69)

Note the previously derived relative velocity natural response term was applied in the derivation of equation (69) for simplicity.

The relative acceleration is

$$
\begin{aligned}
\ddot{z}(t) = &-\omega_n \exp(-\xi\omega_n t)\{\omega_n z(0) + 2\xi\,\dot{z}(0)\}\cos(\omega_d t) \\
&-\frac{\omega_n{}^2}{\omega_d}\exp(-\xi\omega_n t)\left\{-\xi\omega_n z(0) + \left(1-2\xi^2\right)\dot{z}(0)\right\}\sin(\omega_d t) \\
&-\frac{\xi A}{\omega_n{}^2 T}\left\{\xi\omega_n \exp(-\xi\omega_n t)\left[\frac{2\xi}{\omega_n}\cos(\omega_d t) + \frac{1}{\omega_d}\left(2\xi^2 - 1\right)\sin(\omega_d t)\right]\right\} \\
&-\frac{\xi A}{\omega_n{}^2 T}\left\{-\exp(-\xi\omega_n t)\left[\frac{-2\xi\,\omega_d}{\omega_n}\sin(\omega_d t) + \left(2\xi^2 - 1\right)\cos(\omega_d t)\right]\right\} \\
&+\frac{A\omega_d}{\omega_n{}^2 T}\left\{\xi\omega_n \exp(-\xi\omega_n t)\left[\frac{-2\xi}{\omega_n}\sin(\omega_d t) + \frac{1}{\omega_d}\left(2\xi^2 - 1\right)\cos(\omega_d t)\right]\right\} \\
&+\frac{A\omega_d}{\omega_n{}^2 T}\left\{-\exp(-\xi\omega_n t)\left[\frac{-2\xi\,\omega_d}{\omega_n}\cos(\omega_d t) - \left(2\xi^2 - 1\right)\sin(\omega_d t)\right]\right\}
\end{aligned}
$$

$$0 \le t \le T \tag{70}$$

Note the previously derived relative acceleration natural response term was applied in the derivation of equation (70) for simplicity.

The absolute acceleration is

$$\ddot{x}(t) = -\omega_n \exp(-\xi\omega_n t)\{\omega_n z(0) + 2\xi\,\dot{z}(0)\}\cos(\omega_d t)$$
$$-\frac{\omega_n^{\,2}}{\omega_d}\exp(-\xi\omega_n t)\left\{-\xi\omega_n z(0) + \left(1-2\xi^2\right)\dot{z}(0)\right\}\sin(\omega_d t)$$

$$-\frac{\xi A}{\omega_n^{\,2} T}\left\{\xi\omega_n\ \exp(-\xi\omega_n t)\left[\frac{2\xi}{\omega_n}\cos(\omega_d t) + \frac{1}{\omega_d}\left(2\xi^2 - 1\right)\sin(\omega_d t)\right]\right\}$$

$$-\frac{\xi A}{\omega_n^{\,2} T}\left\{-\exp(-\xi\omega_n t)\left[\frac{-2\xi\,\omega_d}{\omega_n}\sin(\omega_d t) + \left(2\xi^2 - 1\right)\cos(\omega_d t)\right]\right\}$$

$$+\frac{A\omega_d}{\omega_n^{\,2} T}\left\{\xi\omega_n\ \exp(-\xi\omega_n t)\left[\frac{-2\xi}{\omega_n}\sin(\omega_d t) + \frac{1}{\omega_d}\left(2\xi^2 - 1\right)\cos(\omega_d t)\right]\right\}$$

$$+\frac{A\omega_d}{\omega_n^{\,2} T}\left\{-\exp(-\xi\omega_n t)\left[\frac{-2\xi\,\omega_d}{\omega_n}\cos(\omega_d t) - \left(2\xi^2 - 1\right)\sin(\omega_d t)\right]\right\} \quad + A\left(\frac{t}{T}\right),$$

$$0 \le t \le T \tag{71}$$

The solution for t > T is the free vibration solution referenced to time T.

As an aside, recall

$$\ddot{z} + 2\xi\omega_n\dot{z} + \omega_n^{\,2} z = -\ddot{y}(t)\ , \qquad 0 \le t \le T \tag{72}$$

$$\ddot{z} = -\ddot{y}(t) - 2\xi\omega_n \dot{z} - \omega_n^{2} z \tag{73}$$

Also,

$$\ddot{x} = \ddot{z} + \ddot{y} \tag{74}$$

Substitute equation (73) into (74).

$$\ddot{x} = -2\xi\omega_n \dot{z} - \omega_n^{2} z \tag{75}$$

Equation (80) is simple for computer programming purposes.

Note that equation (75) is valid for both the base excitation duration and the free vibration duration.

References

1. T. Irvine, Table of Laplace Transforms, Vibrationdata, 1999.
2. T. Irvine, Partial Fractions in Shock and Vibration Analysis, Vibrationdata, 1999.

APPENDIX A

Partial Fraction Expansion

$$\left\{\frac{1}{s^2}\right\}\left\{\frac{1}{s^2+\alpha s+\beta}\right\} = \frac{as+b}{s^2} + \frac{cs+d}{s^2+\alpha s+\beta} \tag{A-1}$$

$$1 = [as+b]\left[s^2+\alpha s+\beta\right] + [cs+d]\left[s^2\right] \tag{A-2}$$

$$1 = \left[as^3+a\alpha s^2+a\beta s\right] + \left[bs^2+b\alpha s+b\beta\right] + \left[cs^3+ds^2\right] \tag{A-3}$$

$$1 = [a+c]s^3 + [a\alpha+b+d]s^2 + [a\beta+b\alpha]s + b\beta \tag{A-4}$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ \alpha & 1 & 0 & 1 \\ \beta & \alpha & 0 & 0 \\ 0 & \beta & 0 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{A-5}$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -\alpha & 1 \\ 0 & \alpha & -\beta & 0 \\ 0 & \beta & 0 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{A-6}$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & \alpha & -\beta & 0 \\ 0 & 1 & -\alpha & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \tag{A-7}$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & -\beta/\alpha & 0 \\ 0 & 1 & -\alpha & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1/\beta \\ 0 \\ 0 \end{bmatrix} \tag{A-8}$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\beta/\alpha & 0 \\ 0 & 0 & -\alpha & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1/\beta \\ -1/\beta \\ -1/\beta \end{bmatrix} \tag{A-9}$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1/\alpha \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1/\beta \\ \alpha/\beta^2 \\ 1/(\alpha\beta) \end{bmatrix} \tag{A-10}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1/\alpha \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} -\alpha/\beta^2 \\ 1/\beta \\ \alpha/\beta^2 \\ -\left[\alpha/\beta^2\right] + \left[1/(\alpha\beta)\right] \end{bmatrix} \tag{A-11}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} -\alpha/\beta^2 \\ 1/\beta \\ \alpha/\beta^2 \\ \left[\alpha^2/\beta^2\right] - [1/\beta] \end{bmatrix} \tag{A-12}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} -\alpha \\ \beta \\ \alpha \\ \alpha^2 - \beta \end{bmatrix}\left[\frac{1}{\beta^2}\right] \tag{A-13}$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -\alpha & 1 \\ 0 & \alpha & -\beta & 0 \\ 0 & \beta & 0 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{A-14}$$

$$\left\{\frac{1}{s^2}\right\}\left\{\frac{1}{s^2 + \alpha s + \beta}\right\} = \left\{\frac{1}{\beta^2}\right\}\left\{\frac{-\alpha s + \beta}{s^2}\right\} + \left\{\frac{1}{\beta^2}\right\}\left\{\frac{\alpha s + \alpha^2 - \beta}{s^2 + \alpha s + \beta}\right\} \tag{A-15}$$