Smallwood
Damped Sinusoids

pg 26-27 low freq
Compensation

(Part 2 of 2 Parts)

Reprinted from

SEMINAR ON UNDERSTANDING DIGITAL CONTROL AND ANALYSIS IN VIBRATION TEST SYSTEMS

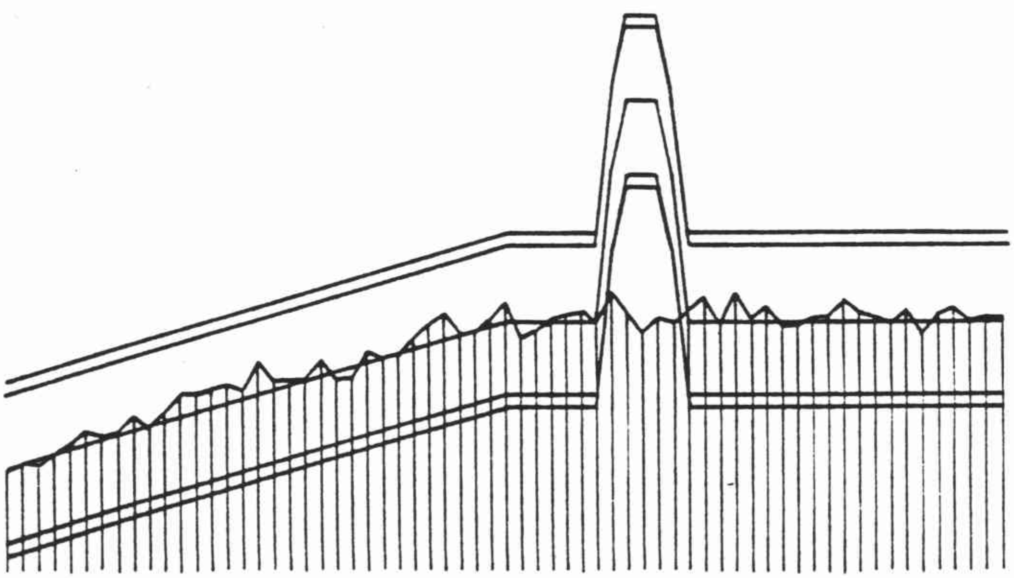
≈ 1976

Jointly Sponsored By

GODDARD SPACE FLIGHT CENTER
Greenbelt, Maryland

JET PROPULSION LABORATORY
Pasadena, California

THE SHOCK AND VIBRATION INFORMATION CENTER
Naval Research Laboratory, Washington, D.C.



A Publication of
THE SHOCK AND VIBRATION
INFORMATION CENTER
Naval Research Laboratory, Washington, D.C.

TIME HISTORY SYNTHESIS FOR SHOCK TESTING ON SHAKERS

D. O. Smallwood
Sandia Laboratories
Albuquerque, New Mexico

Several digital methods now available for matching shock spectra with oscillatory type transients are reviewed and the merits of each discussed. The methods include WAVSYN, SHOC, sums of decaying sinusoids, fast-sine sweeps, modulated pseudo random noise, classical pulses, and the method of least favorable response. While the last method is not strictly a method for matching shock spectra, it is closely related. The discussion is limited to procedures for synthesizing the waveform. Additional parameters (in addition to the shock spectrum) which can be specified to limit the classes of functions available for performing a particular test are discussed.

INTRODUCTION

The need for an oscillatory-type pulse to represent certain field environments has been recognized for several years. During this period, it was also recognized that electrodynamic or electrohydraulic shaker systems were well-suited for reproducing these pulses. With the development of digital control techniques it became possible to reproduce very complex time histories. Therefore, considerable effort has been expended in developing these methods.

A common tool for measuring the character of a transient has been the shock response spectrum of the event. As a result, many current methods are based on producing a transient whose shock spectrum matches a specified shock spectrum. Although considerable controversy still exists regarding the value of the shock spectrum, its use is very common and will continue for years regardless of the outcome of the controversy. Because of its projected use and the fact that procedures which exist can result in quite different waveforms, it was felt that a review of the available techniques would be useful.

Both analog and digital methods are being used. However, the purpose of this paper is to review the digital techniques. The problem addressed in this paper is: given a specified shock spectrum how can a waveform be generated which will have the same (within some tolerance) shock spectrum. The companion problem of how to reproduce the synthesized time history is left for another paper.

Before the time history synthesis can be discussed, the limitations placed on the waveform by the shaker system will be reviewed.

The types of transient vibration or shock pulses which can be accurately reproduced on both electrodynamic and electrohydraulic exciters are very much dependent on the physical limitations of the exciters. These limitations are listed in Table 1 and are briefly discussed here.

TABLE 1
Exciter Limitations

Limitation Number	Initial	Final	Maximum
1	$\ddot{x}(0) = 0$	$\ddot{x}(T) = 0$	Limited
2	$\dot{x}(0) = 0$	$\dot{x}(T) = 0$	Limited
3	$x(0) = 0$	$x(T) = 0$ (Electrodynamic)	Limited

The initial and final acceleration and velocity of a transient must be zero for both electrodynamic and electrohydraulic exciters. As with any type of testing machine, maximum attainable values of acceleration and velocity are limited. Acceleration is actually limited by the force capabilities of the exciter.

Flexures in electrodynamic exciters generate restoring forces which return the exciter table to its originating position (defined as zero), and limitation 3 holds. This is not a requirement for electrohydraulic systems; however, by imposing this limitation one can take advantage of both the forward and return portion of the stroke to generate the required transient. This effectively results in doubling the displacement capacity of the exciter for generating transients. Thus for the purposes of this discussion limitation 3 will also be considered a limitation for an electrohydraulic exciter.

The initial slope of a lightly damped shock spectrum is related to the velocity and displacement changes required. This is illustrated in Figure 1. Here is an example of waveforms for different values of the initial slope (i.e., the slope of the shock spectrum as the frequency approaches zero on a log-log plot). Note that the shaker limitations will require that only pulses of the last type M>12) can be reproduced on a shaker system. If the initial slope of the specified shock spectrum is not greater than 12 dB/octave there will exist a lower frequency below which the shaker displacement and velocity limits will not permit the spectrum to be matched. If the test item responds like a rigid body at all frequencies less than this lower limit an argument can be made that it is not important to match the spectrum in this range.

Techniques using oscillatory pulses whose shock spectra have an initial slope of S = 1, as for example decaying sinusoids, have been moderately successful because the shaker system will act as a high-pass filter removing the low frequency energy from the waveform. This, combined with the flexure restoring force, will force the velocity and displacement to return to zero. The manner in which the acceleration-time waveform is distorted to remove the velocity and displacement change will be characteristic of the individual shaker system used. This makes it difficult to predict the velocity and displacement waveforms until after the test is run. The velocity and displacement waveforms

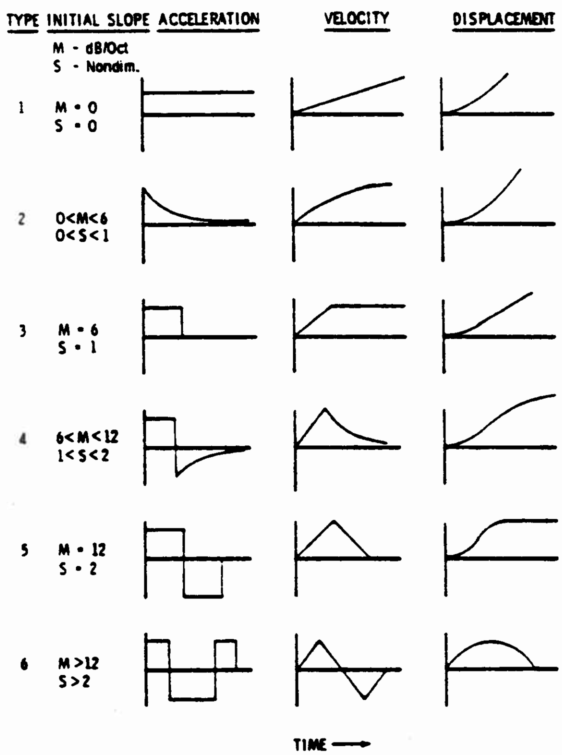


Fig. 1 - Initial Slope of the Shock Spectrum for Some Common Waveforms

are important as the shaker system places limits on the magnitudes which can be reproduced.

The shock spectrum at very high frequencies is identical to the peak amplitude of the input time history. Again the shaker limitations will determine if this level can be reproduced. In many real specifications the shock spectrum is not specified to a high enough frequency to determine the required peak input. In this case some flexibility is available in synthesizing the pulse.

With this introduction to the limitations of shaker systems and to the characteristics of a shock spectrum, the methods used to generate a time history whose shock spectrum will match a specified spectrum can be discussed.

PARALLEL FILTER METHODS

The first method discussed is the parallel filter method. This technique is a direct digital implementation of older analog methods [1]. Using this technique the waveform synthesis and shaker equalization are not separated but accomplished together with the test item mounted on the shaker. Using this method the required shock spectrum is broken into regions (frequency ranges, typically 1/3 octaves). For each region the peak response is specified, along with a basic waveform whose energy is concentrated in the same frequency range. Any of the basic waveforms discussed later could be used. The most common ones are WAVSYN and decaying sinusoids. In the earlier analog methods the basic waveform was the response of a bandpass filter to an impulse. An initial amplitude for each basic waveform (one for each region or frequency band) is chosen and a composite waveform is formed by summing the waveforms. This time history is then applied to the power amplifier and the control accelerometer response is measured. The shock spectrum of the response is calculated and compared with the required spectrum. This information is then used to modify the amplitudes of the basic waveforms and the process is repeated as illustrated in Figure 2.

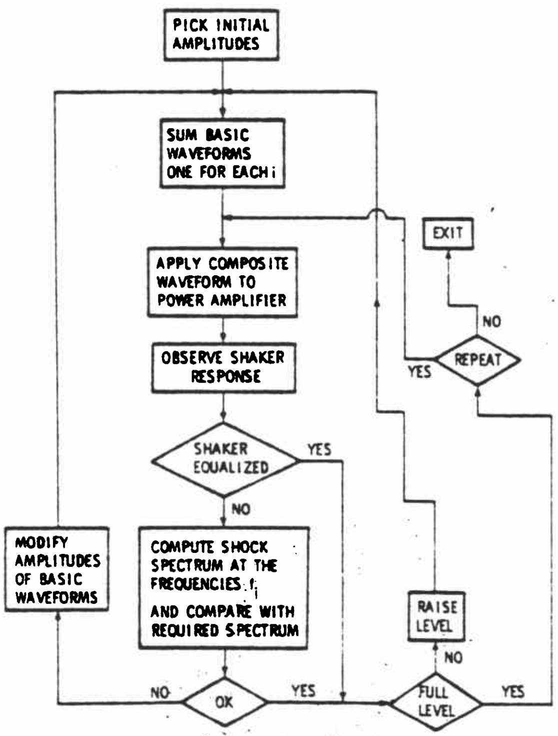


Fig. 2 - A Flow Chart for a Parallel Filter Iteration Method

A variation of the procedure outlined in Figure 2 is shown in Figure 3. The principal difference between this variation and the previous one is that the frequency components are added one at a time starting with the lowest frequency and working up to the highest frequency. When all the components have been added the complete spectrum is

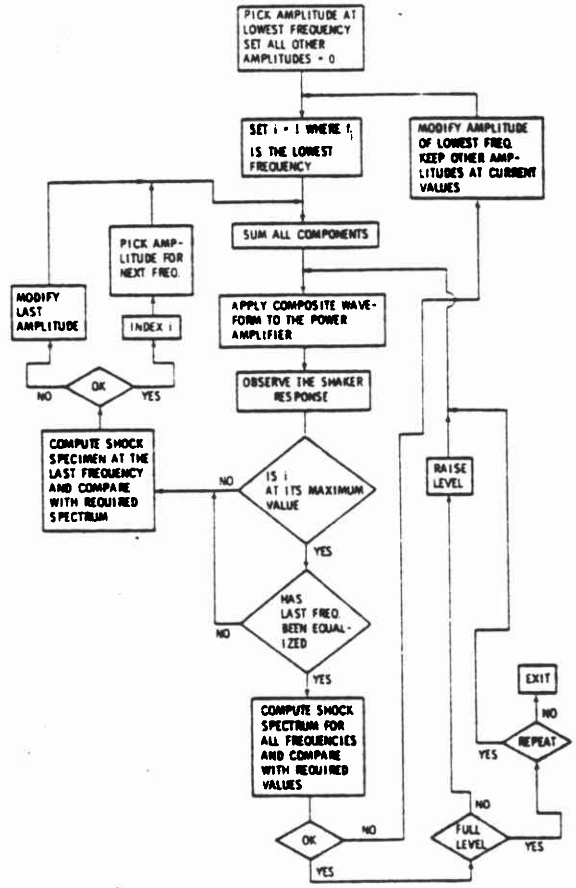


Fig. 3 - A Modified Flow Chart for a Parallel Filter Iteration Method

checked. If the spectrum is not within specifications the procedure is started over at the lowest frequency using the previously determined amplitudes as starting values. Using this technique only one frequency component at a time is modified. As a result, the method is more stable than the previous method (more likely to converge) as the interaction between components is less likely to cause problems. The advantage of the previous method is that if a stable solution is found, fewer pulses are required to equalize the system.

The only difference between the digital implementation and the older analog methods is that a larger variety of basic waveforms can be used, the shock spectrum is computed digitally, and more sophisticated and automated amplitude modification methods can be used. The advantages of these methods include: The methods are relatively easy to implement and use, as details of the time history are not controlled. Quite large values of the shock response spectrum can be generated as the shaker is not required to be equalized over sharp notches in the frequency response function. This tends to concentrate the pulse energy in those frequencies where the shaker responds well. Also since equalization is good only in a broad sense, a large variety of shock spectra can be matched. The disadvantages include: Since details of the time history are not controlled, little is known about the velocity and displacement requirements. Because the shaker is not equalized with a fine frequency resolution, all frequencies may not be adequately tested. If the shock spectrum is analyzed with a finer resolution than the initial range, deviations from the required spectrum may be found.

TIME HISTORY SYNTHESIS

Several methods have been developed to synthesize a time history which will match a specified shock spectrum. The synthesis is usually done on a digital computer. It is assumed that this time history can be reproduced on a shaker system using methods originally developed by Favour and LeBrun [2] and will not be discussed in this paper. Several of the methods have checks included to determine the suitability of the transient for shaker reproduction. Reproducing the pulse on the shaker system is independent of the procedure to develop the waveform.

One method will be discussed in detail. The remaining methods will be discussed more briefly as the procedures are similar. The first method discussed is sums of decaying sinusoids (DS). The other methods discussed include WAVSYN, Shaker Optimized Cosines (SHOC), fast-sine sweeps, modulated random noise, modification of field time histories, classical pulses, and least favorable responses.

Sums of Decaying Sinusoids

It has been recognized for years that many field environments can be adequately represented by sums of decaying sinusoids, and as a result several authors [3, 4, 5] have suggested their use to match shock spectra. The usual basic waveform is given by

$$g_m(t) = A_m e^{-\zeta_m \omega_m t} \sin \omega_m t \qquad t \geq 0$$
$$= 0 \qquad t < 0$$

The basic waveform is shown as Figure 4, and the normalized shock spectra, for several values of ζ are shown as Figure 5. Several waveforms are added to synthesize a time history to match a complex shock spectra.

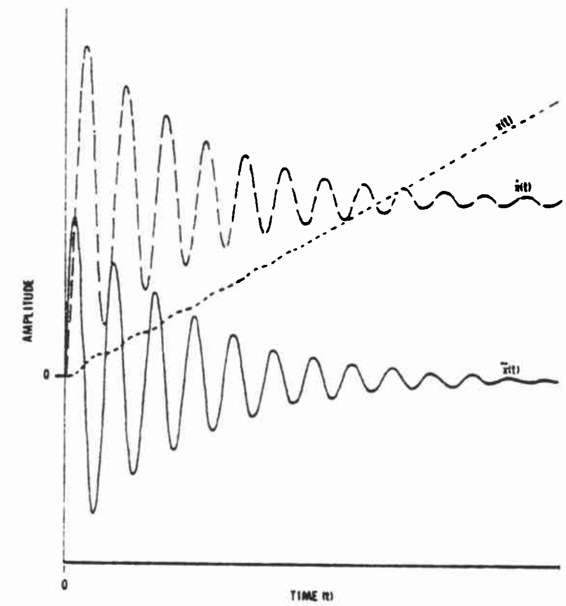


Fig. 4 - Decaying Sinusoid Acceleration, Velocity and Displacement Characteristics

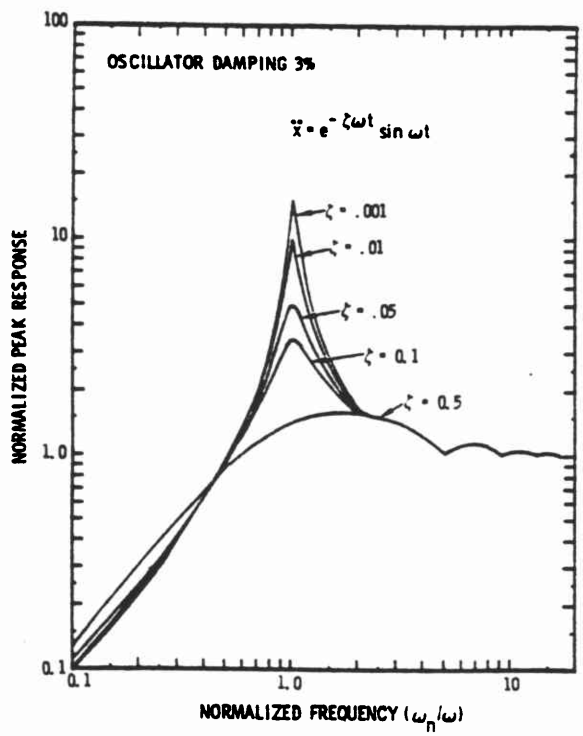


Fig. 5 - Normalized Shock Spectra for a Single Frequency Decaying Sinusoid

However, this waveform does not meet the requirements of a zero velocity and displacement change. If an attempt is made to reproduce this waveform, it will be slightly distorted by the shaker system removing the velocity and displacement change. Since the exact distortion will be a function of the shaker system, it becomes difficult to predict the velocity and displacement waveforms.

Two modifications of the basic waveform have been suggested. The author [3] suggested that a time-shifted, highly damped decaying sinusoid be added for velocity and displacement compensation. It is also possible to introduce time shifts to improve the appearance of the waveform. Allowing for time shifts of the individual components, the waveform becomes

$$\ddot{x}(t) = \sum_{m=1}^{L} U(t-\tau_m) A_m e^{-\zeta_m \omega_m (t-\tau_m)} \cdot \sin \omega_m (t-\tau_m)$$

$$+ U(t+\tau)\, A_c e^{-\zeta_c \omega_c (t+\tau)} \cdot \sin \omega_c (t+\tau)$$

where

$$A_c = -\omega_c(\zeta_c^2 + 1) \sum_{m=1}^{L} \frac{A_m}{\omega_m(\zeta_m^2 + 1)}$$

$$\tau = \frac{\omega_c(\zeta_c^2 + 1)}{A_c} \left\{ \frac{2\zeta_c A_c}{\omega_c^2(\zeta_c^2+1)^2} + \sum_{m=1}^{L} \left[\frac{A_m \tau_m}{\omega_m(\zeta_m^2+1)} + \frac{2\zeta_m A_m}{\omega_m^2 (\zeta_m^2 + 1)^2} \right] \right\}$$

The magnitude (A_c) and the shift (τ) of the velocity and displacement compensating pulse are fixed by the other parameters. A plot of a typical waveform is shown as Figure 6.

Prasthofer and Nelson [4] suggested velocity and displacement compensation by adding two exponential pulses and a phase shift to the decaying sinusoid to give

$$\ddot{x}(t) = \sum_{m=1}^{L} \ddot{x}_m(t)$$

$$\ddot{x}_m(t) = A_m \left\{ K_1 e^{-at} - K_2 e^{-bt} + K_3 e^{-ct} \sin(\omega_m t + \theta) \right\}$$

where

$$\omega_c = \omega_m / \sqrt{1-\zeta_m^2}$$

$$a = \omega_c / 2\pi$$

$$b = 2\zeta_m \omega_c$$

$$c = \zeta_m \omega_c$$

$$K_1 = \frac{\omega_m a^2}{(a-b)\left[(c-a)^2 + \omega_m^2\right]}$$

$$K_2 = \frac{\omega_m b^2}{(a-b)\left[(c-b)^2 + \omega_m^2\right]}$$

$$K_3 = \left\{ \frac{(c^2-\omega_m^2)^2 + 4c^2\omega_m^2}{\left[(b-c)^2 + \omega_m^2\right]\left[(a-c)^2 + \omega_m^2\right]} \right\}^{1/2}$$

$$\theta = \tan^{-1} \frac{-2c\omega_m}{c^2-\omega_m^2} - \tan^{-1} \frac{\omega_m}{a-c} - \tan^{-1} \frac{\omega_m}{(b-c)}$$

The first two terms are added for velocity and displacement compensation. The phase shift, θ, is added to force the initial value to zero.

A plot of a typical modified pulse is shown as Figure 7.

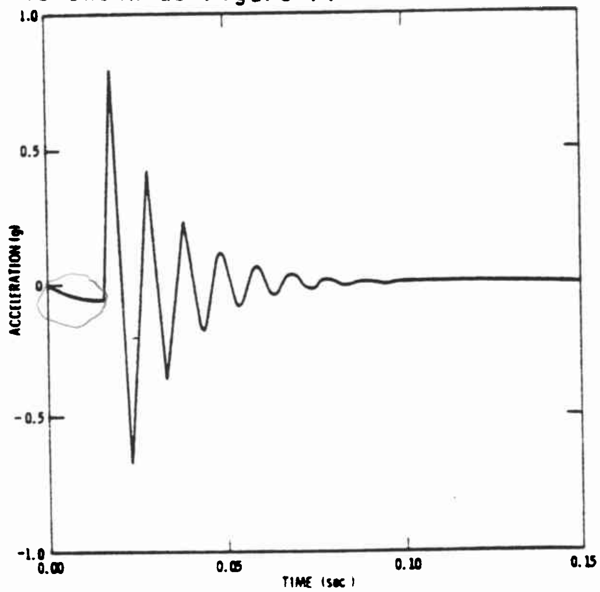


Fig. 6 - Modified Decaying Sinusoid

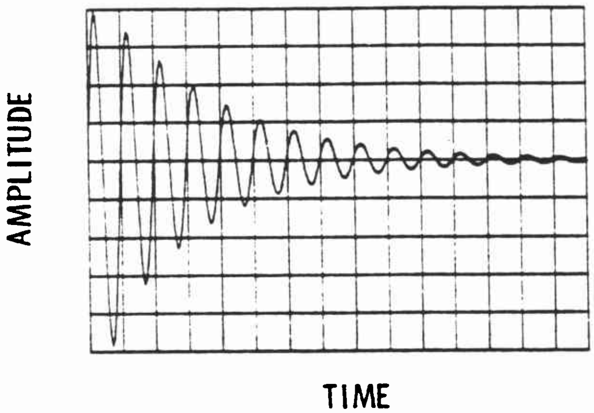


Fig. 7 - Modified Decaying Sinusoid

A set of curves can also be generated which gives the peak response ratios (the ratio of the peak response to the input amplitude at the frequency of highest response) as a function of the decay rate (ζ) of the decaying sinusoid and the damping factor (η) of the shock spectrum (Figure 8).

Example Using Decaying Sinusoids

Let us assume a test requires that a pulse be reproduced on an electro-hydraulic shaker system. The required shock spectrum is given in Figure 9. A typical field time history which resulted in the required composite shock spectrum is also included (Figure 10) for comparison purposes. The shaker limitations are: peak acceleration 50 g's o-p, peak velocity 120 in/sec o-p, and peak displacement 8 in p-p. The pulse will be composed of sums of decaying sinusoids compensated using the method suggested by the author.

The spectrum cannot be matched at the lowest frequency because the slope of the spectrum is less than 12 db/oct. In actual fact, a 16 ft/sec velocity change is associated with the spectrum. It will be assumed that the test item is not likely to have any resonances below 6 Hz or above 115 Hz and the spectrum will be matched from 4 Hz to 115 Hz. The first step is to assume all the decaying sinusoids will act independently. Using Figures 5 and 8 the amplitudes and decay rates for the components can be picked. A 4 Hz pulse is picked to raise the low end of the spectrum. A large decay rate is picked to keep the spectrum reasonably flat in this region. Further frequencies are picked at the peaks in the spectrum.

The initial estimates (Trial 0) are shown in Figure 11 and Table 2. From displacement considerations the compensating frequency is picked at 1 Hz with a decay rate of 1.0. The shock spectrum of the composite pulse is shown on Figure 12. The spectrum is low at 4 Hz due to the interaction of the 1 Hz pulse. The spectrum is also low from 4-12 Hz, high from 12-90 Hz, and low above 90 Hz. As expected the assumption of independence was only approximate. An iterative procedure is then used to modify the component amplitudes to match the spectrum at the frequencies of the decaying

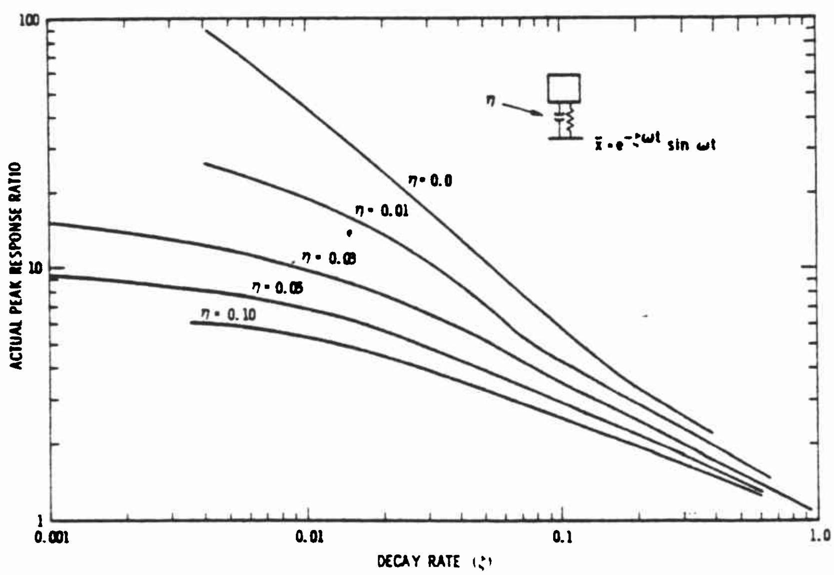


Fig. 8 - Peak Response Ratio for a Decaying Sinusoid

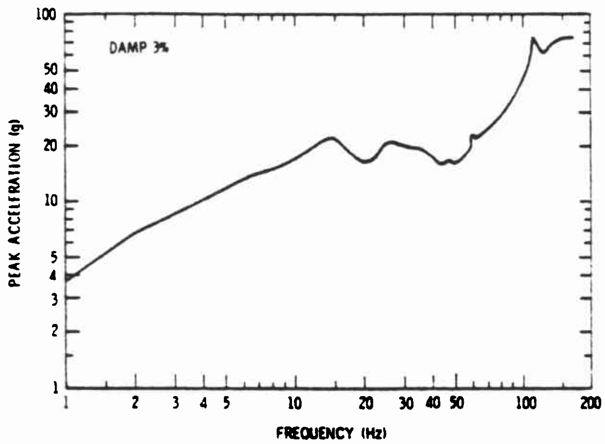


Fig. 9 - Example Specified Shock Spectrum

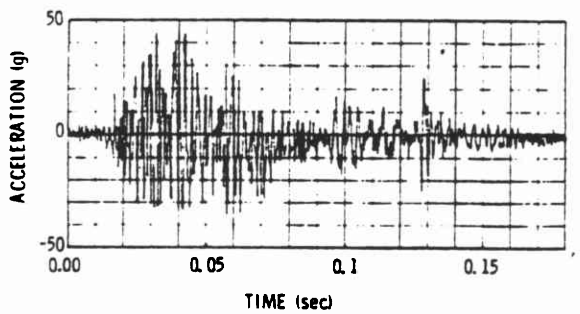


Fig. 10 - A Sample Time History

sinusoids. One possible iteration procedure is outlined in Figure 13. The results are shown in Figures 12 and Table 3 as Trial 1. Note that the solution does not converge; i.e., the amplitudes of the 36 and 47 Hz components were reduced to zero, but the spectrum was still too high. The high level is caused by the interaction of the lower frequency components. For the next trial (Trial 2), the decay rates of the 26 Hz, 36 Hz, 47 Hz, and 60 Hz components were lowered to reduce the interaction. Components at 8 Hz and 80 Hz were added to raise the level near those frequencies. The level of the shock spectrum at 112 Hz was also raised to improve the match at that frequency. Again an iteration procedure was used to determine the component amplitudes which would result in matching the required spectrum at the component frequencies. As before the amplitude of the 47 Hz component was reduced to zero. This time the spectrum was about 5% high at 47 Hz. The results are shown in Table 4 and Figure 12. At this point it was determined that the match was sufficient and the pulse parameters were within the shaker capabilities.

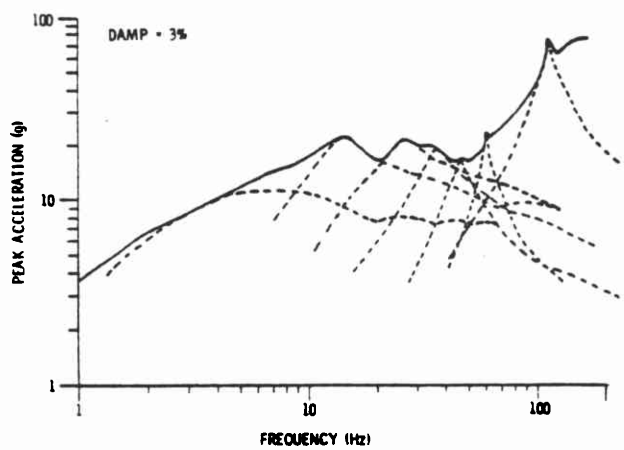


Fig. 11 - Composite Shock Spectrum Formed by Several Decaying Sinusoids

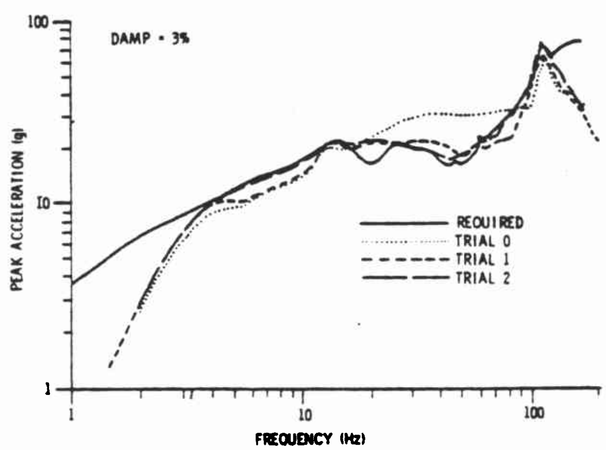


Fig. 12 - Shock Spectrum for Several Composite Waveforms Composed of Sums of Decaying Sinusoids

In this example an acceptable match was achieved in two iterations (Trial 0 is not counted, as it is usually skipped, but was included for illustrative purposes.) The time histories of the acceleration, velocity and displacement are included as Figures 14-16. The parameters in Table 4 can now be used to define a time history to be reproduced on the shaker system.

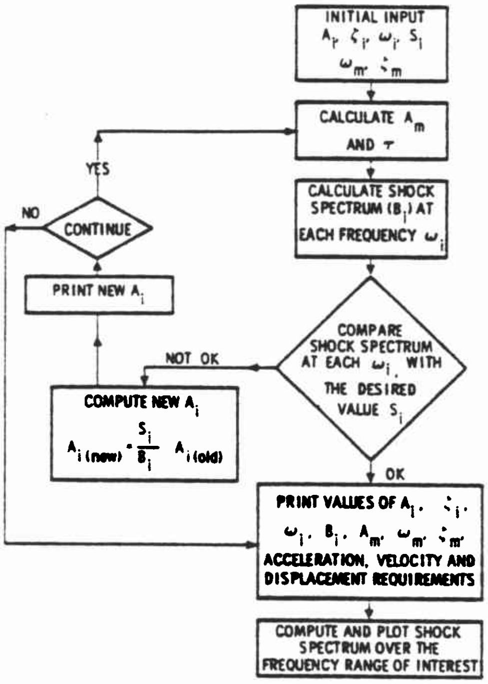


Fig. 13 - Flow Diagram for Picking Decaying Sinusoids to Match a Given Shock Spectrum

TABLE 2 - Parameters for Trial 0

I	FREQ (Hz)	ZETA	AMP (g)	TAU (sec)
1	1	1.0	-5.37	-0.141
2	4	0.5	7.1	0
3	14	0.2	8.8	0
4	26	0.2	8.8	0
5	36	0.1	5.7	0
6	47	.05	3.4	0
7	60	.01	2.3	0
8	112	.02	8.1	0

TABLE 3 - Parameters for Trial 1

I	FREQ (Hz)	ZETA	AMP (g)	TAU (sec)
1	1	1.0	-5.71	-0.138
2	4	0.5	8.67	0
3	14	0.2	10.1	0
4	26	0.2	8.52	0
5	36	0.1	0	0
6	47	.05	0	0
7	60	.01	2.22	0
8	112	.02	8.47	0

TABLE 4 - Parameters for Trial 2

I	FREQ (Hz)	ZETA	AMP (g)	TAU (sec)
1	1	1	-5.74	-0.141
2	4	0.5	6.47	0
3	8	0.3	4.45	0
4	14	0.2	11.1	0
5	26	.08	3.51	0
6	36	.05	0.646	0
7	47	.03	0	0
8	60	.005	1.78	0
9	80	.01	2.73	0
10	112	.02	9.93	0
Acceleration Range			+18.1, -9.0 g	
Velocity Range			+63, -75 in/s	
Displacement Range			+2.1, -4.9 in	

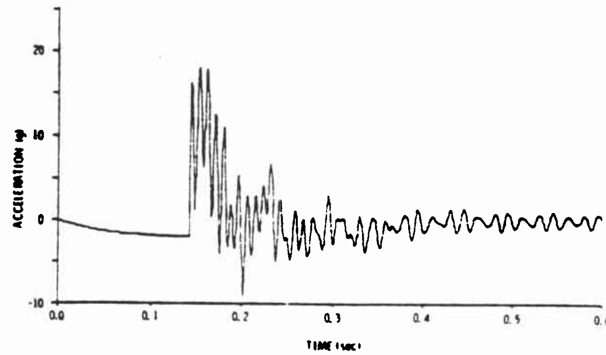


Fig. 14 - Acceleration of a Composite (Trial 2) Composed of a Sum of Decaying Sinusoids

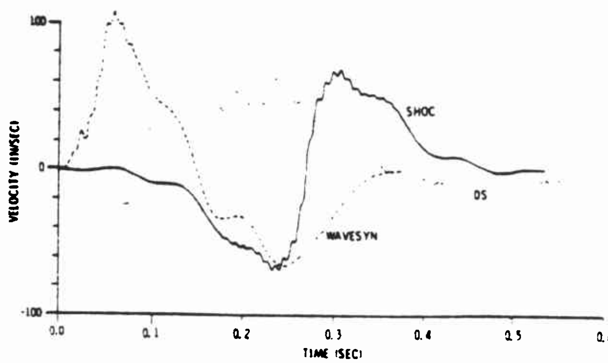


Fig. 15 - Velocity of Three Composite Waveforms

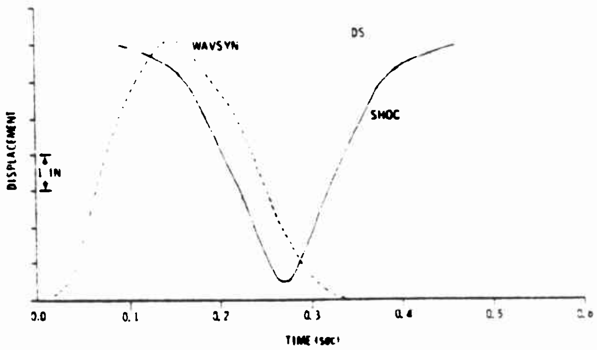


Fig. 16 - Displacement of Three Composite Waveforms

WAVSYN

A method developed for the U. S. Army is described by Yang. [6, 7] This method is called WAVSYN, The basic waveform is given by

$$g_m(t) = A_m \sin(2\pi b_m t)\sin(2\pi f_m t) \quad \text{for } 0 \le t \le T_m,$$

$$= 0 \quad \text{for } t > T_m,$$

where $f_m = N_m b_m$,

$T_m = 1/(2b_m)$,

and N_m is an odd integer.

The waveform for $N_m = 5$ is shown as Figure 17.

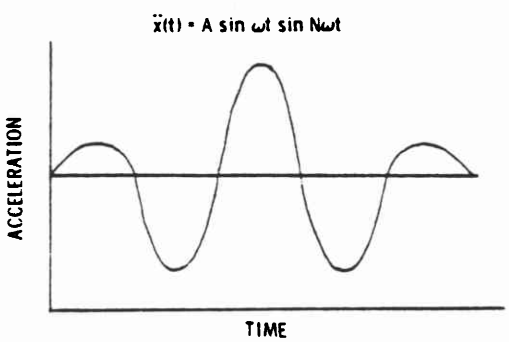


Fig. 17 - WAVSYN Time History, $N_m = 5$

A specified shock spectrum is then matched by summing several of the basic waveforms.

$$\ddot{x}(t) = \sum_{m=1}^{L} g_m(t + \tau_m).$$

The basic adjustable parameters are the number of pulses summed L, the frequencies f_m, the number of half cycles N_m, and the amplitudes of each component A_m.

The delay τ_m is used to improve the appearance of the time history (i.e., make the time history look more like field data) and does not seriously change the shock spectrum. Normalized shock spectra for N = 3, 5, 7, and 9 are shown in Figure 18. Note that the peak in the shock spectra occurs near f_m.

The shape or magnification can be modified by changing N_m. The frequency of the peak in the shock spectrum can be controlled with f_m and the magnitude of the whole curve can be moved up and down with A_m. Detailed procedures [6, 7] are described for matching a wide variety of shock spectra using the technique. An advantage of the technique is that the velocity and the displacement changes are zero. This is essential for accurate reproduction on a shaker system.

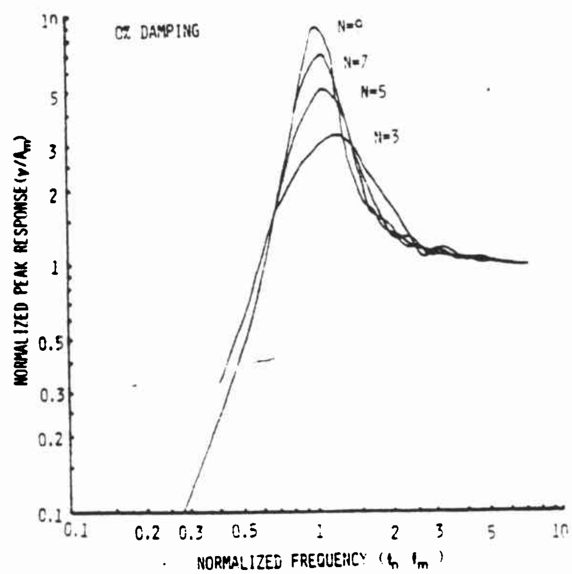


Fig. 18 - WAVSYN Shock Spectra

For comparison, the same shock spectrum as in the previous example was matched using WAVSYN. Eight components were used as listed in Table 5.

TABLE 5 - WAVSYN Components

No	f_m(Hz)	N_m	A_m(g)	τ_m
1	4	3	3.98	0
2	6	3	4.82	0
3	9	3	7.37	0
4	12.5	7	3.	0
5	27	5	4.3	0
6	33	7	0.94	0
7	70	11	2.44	0
8	112	19	8.86	0
Maximum Acceleration 20.9 g's				
Maximum Velocity 107 in/sec				
Maximum Displacement 7.10 in				

The acceleration, velocity, and displacement time histories are shown in Figures 19, 15, and 16. The shock spectrum is shown in Figure 20.

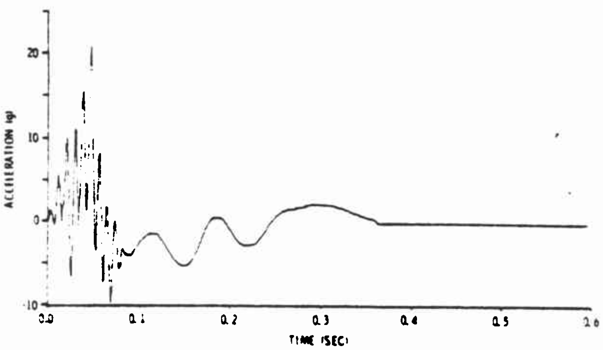


Fig. 19 - Acceleration Time History of WAVSYN Composite Pulse

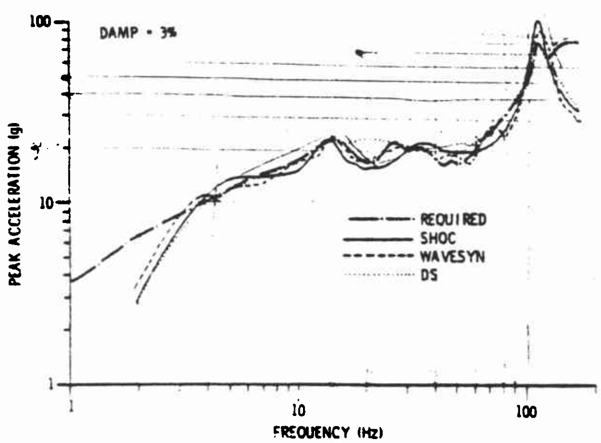


Fig. 20 - Shock Spectra Comparing Decaying Sinusoids, SHOC, and WAVSYN

SHOC

A method developed by the author [8] for matching shock spectra with an oscillating transient is called the SHOC (Shaker Optimized Cosines) technique. The basic waveform for this technique is given by

$$g_m(t) = A_m e^{-\zeta_m \omega_m t} \cos\omega_m t - B_m \cos^2 \frac{\pi t}{\tau} \qquad 0 \le t \le \tau/2$$

$$= 0 \qquad t > \tau/2$$

$$g_m(-t) = g_m(t),$$

$$\text{where } B_m = \frac{4 A_m \zeta_m}{\tau\omega_m(\zeta_m^2+1)},$$

τ = the pulse duration

The first term is an exponential decaying cosine and the second term (a cosine bell) is added to force the velocity and the displacement changes to zero. The waveform is symmetric about the origin and builds up and then decays.

The basic waveform is illustrated as Figure 21. The normalized shock spectra for several values of ζ are shown as Figure 22. As in the case of WAVSYN, a parameter is available for modifying the shape of the curve (ζ_m), the frequency of the peak of the shock spectrum (ω_m) and the level of the curve (A_m).

As in the case of WAVSYN, several of these components are added to synthesize a waveform which will match a complex shock spectrum

$$\ddot{x}(t) = \sum_{m=1}^{L} g_m(t)$$

Also, as in WAVSYN, the velocity and the displacement changes are zero.

For comparison, the same shock spectrum as used in the two previous examples was matched using SHOC. Four components were used and are listed in Table 6.

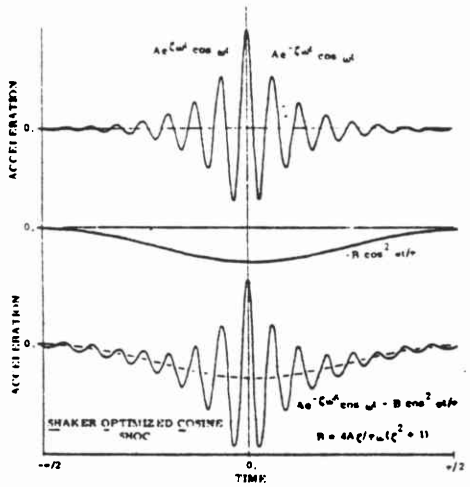


Fig. 21 - A SHOC Pulse

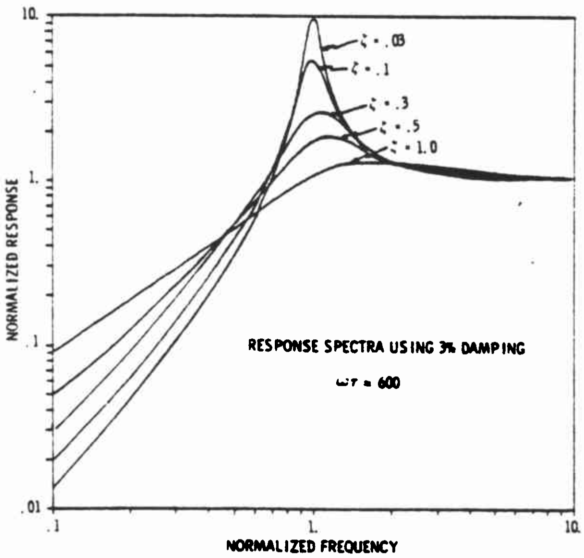


Fig. 22 - SHOC Shock Spectrum

TABLE 6 - SHOC Components

FREQ (Hz)	ζ	Amplitude	τ(sec)
4.0	1.0	10.0	0.54
13.0	0.1	2.5	
32.0	0.2	3.0	
112.0	.03	9.8	
Maximum Acceleration		23.7	
Maximum Velocity		69.5	in/sec
Maximum Displacement		6.5	in

The acceleration, velocity, and displacement time histories are shown in Figures 23, 15, and 16. The shock spectrum is shown in Figure 20. The three previous methods are also compared in Table 7.

Note that for this example the characteristics of all three pulses are quite similar, and the three methods would be expected to have a similar damage potential, although this has not been firmly established.

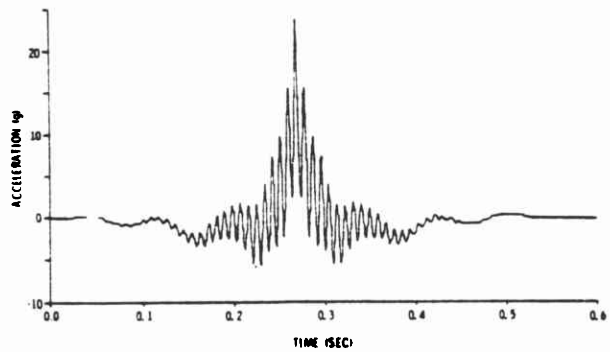


Fig. 23 - Acceleration of a Composite SHOC Pulse

Fast-Sine Sweeps

The first analytical procedure developed for matching shock spectra was a technique using fast-sine sweeps, which lasted from a few hundred milliseconds to several seconds. The early work in this area was done by Crum and Grant [9] and later expanded by several authors including Roundtree and Freberg [10]. The basic procedure is to assume an acceleration time history of the form

$$\ddot{X}(t) = A(t) \sin \theta(t)$$

A particular form is then chosen for for A(t) and θ(t) which includes a limited number of variable parameters. The derivative of θ(t) is the instantaneous frequency of the time history. Procedures are then given for picking the variable parameters to match the specified spectrum. Crum and Grant chose

$$A(t) = A \quad \text{(a constant)}$$

$$\theta(t) = -2\pi N' \ln\left(1 - \frac{f_o t}{N'}\right) \quad \text{increasing frequency}$$

$$\theta(t) = -2\pi N' \ln\left(1 + \frac{f_o t}{N'}\right) \quad \text{decreasing frequency}$$

TABLE 7 - Comparison of Decaying Sinusoids, WAVSYN, and SHOC

Method	Peak Input (G)	Duration[1] (ms)	RMS Duration[2] (ms)	Ratio of Peak Response to the Peak Input	Velocity Requirement (in/sec)	Displacement Requirement (in)
WAVSYN	21	300	71	4.3	110	7.1
SHOC	24	120	40	4.0	70	6.5
Sums of Decaying Sinusoids	19	210	88	4.7	70	6.9

(1) The duration is defined as the time beginning when the waveform first reaches 10-percent of its peak value and ending when it decays to 10-percent of its peak value for the last time.

(2) See Appendix A

where the variable parameters are the constant for A(t), the starting frequency f_0, and the effective number of cycles at each frequency N'. The effective number of cycles at each frequency (N') is based on the ratio of the shock spectra with a Q of 5 and 25, f_0 was chosen as the lowest frequency of interest, and A was determined by the required amplitude of the shock spectrum. The form chosen by Roundtree and Freberg was a more general formulation given by

$$\frac{d\,(\ln A(t))}{d\,(\ln f(t))} = \beta,\ A(0) = a,\ f(0) = f_o,$$

$$\frac{df(t)}{dt} = Rf(t)^{\gamma},$$

$$\frac{d\theta(t)}{dt} = 2\pi f(t),\ \theta(0)=0,$$

where the variable parameters are β, a, f_o, R, γ.

As mentioned earlier, the derivative of θ(t) is the instantaneous frequency f(t). β represents the rate of change of the amplitude function, A(t), with respect to the instantaneous frequency, f(t). The starting frequency is f_0. The initial value for the amplitude function A(t) is a. R and γ are used to control the rate of frequency change and the sweep duration. For example, if β = 0, the sweep amplitude is held constant, and if γ = 0, the sweep rate will be linear. The sweep will be exponential for γ = 2.

Roundtree and Freberg present graphs for picking values for the parameters based on the requested shock spectra computed at two different values of damping.

The methods of Crum and Grant have been used for several years. The methods of Roundtree and Freberg are too new to be well established. The advantage of these methods is that they produce pulses which are well suited for reproduction on shaker systems. It has also proved practical to match shock spectra at two different values of damping. The principle disadvantage is that the techniques produce pulses which do not resemble (in the time domain) many (if not most) field events. Another limitation is that the techniques have been highly developed for matching spectra which can be represented by a straight line on a log-log shock spectrum plot and adapt poorly for matching spectra with other shapes. A typical time history for these methods is shown in Figure 24.

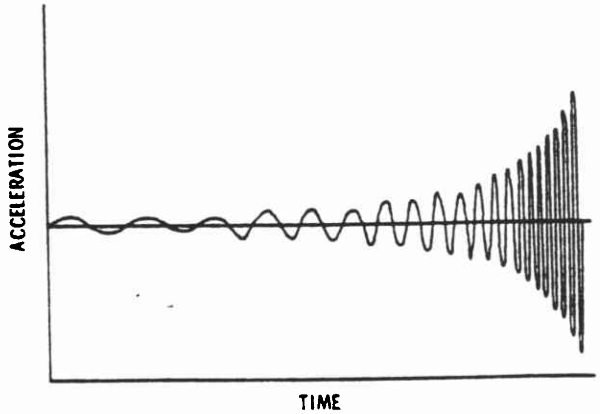


Fig. 24 - Time History of a Fast-Sine Sweep

Modulated Random Noise

Authors in the seismic field have long recognized the somewhat random nature of seismic records. As a result, numerous proposals [11, 12] have been made to derive a random process which, when multiplied by a suitable window, will provide an adequate simulation of seismic events. The attempt is to derive a waveform which will have the statistical characteristics of a seismic event. Since one of the tools used to characterize the seismic event is the shock spectrum of the event, the modulated random noise methods produce pulses which match a specified shock spectrum. However, it is important to recognize that these methods only match the shock spectrum in a probabilistic sense.

Bucciarelli and Askinaza [13] proposed that these same ideas could be used to simulate pyrotechnic shocks by using an exponential window.

$$\ddot{x}(t) = g(t)n(t)$$

where g(t) is a deterministic function of time which characterizes the transient nature of the event.

$$g(t) = \begin{cases} 0 & t<0 \\ e^{-\beta t} & t\geq 0 \end{cases}$$

The function n(t) is a representative of a stationary, broadband, zero-mean, noise process with a spectral density $S_n(\omega)$.

It is then proposed that the function $S_n(\omega)$ be chosen such that the average Fourier amplitude spectrum of $\ddot{x}(t)$ will be equal to the average Fourier amplitude spectrum of the field environment. It is shown that the following approximation will accomplish this purpose.

$$E[F(\omega)F^*(\omega)] \cong S_n(\omega)/2\beta$$

where

$$E[F(\omega)F^*(\omega)]$$

is the expected value of the field Fourier amplitude spectrum.

The above equations seem reasonable for the following reason. $S_n(\omega)$ is a stationary process with a fixed amount of energy per unit of time. As β becomes larger, the duration of the transient becomes shorter and $S_n(\omega)$ must be increased to maintain a given energy in the transient.

Tsai [14] then proposes that the random variation in the shock spectrum be removed in the following manner. Pick a sample $\ddot{x}(t)$. The shock spectrum of $\ddot{x}(t)$ is then computed. In areas where the shock spectrum is low, add energy to the waveform by adding sine waves to n(t). In areas where the shock spectrum is high, filter n(t) with a narrow-band notch filter. Compute the shock spectrum of the modified $\ddot{x}(t)$ and repeat the process until the desired shock spectrum is achieved. The final modified $\ddot{x}(t)$ will then be used as the test input. A procedure could probably be worked out to accomplish the same goal by modifying the Fourier amplitude of n(t) without resorting to adding sine waves and filtering [15,16].

This procedure looks quite interesting, but as far as the author knows, it has not been tried in the laboratory.

Modification of Field Time History

Workers in the seismic field have suggested a method very similar to the modulated random noise technique. For this method the original sample (x(t)) is an actual field time history (in the seismic field an earthquake record). The Fourier content of the waveform is then modified to force the shock spectrum of the time history to match a specified curve.

Classical Pulses

It may be desirable to reproduce a classical pulse (half-sine, terminal peak sawtooth, etc.) on a shaker system. In general these pulses are of Type 3 (Fig. 1) and cannot be reproduced without modification. The pulses can be easily corrected to a Type 5 pulse by adding a bias pulse of equal and opposite area at the end of the pulse. Pulses of this type can be reproduced with only the distortion introduced by the improper displacement boundary conditions. It is always possible to introduce a bias pulse which will also meet the displacement boundary conditions. For example, consider a terminal peak sawtooth pulse biased with a square wave as shown in Figure 25.

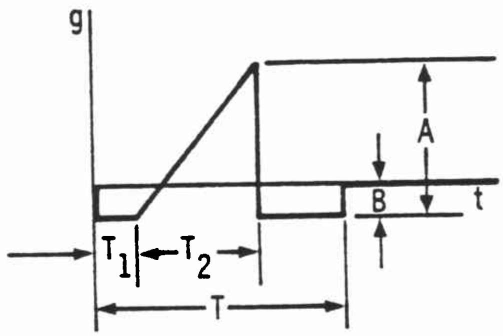


Fig. 25 - A Modified Terminal Peak Sawtooth

It is easy to see that the velocity change will be zero if

$$BT = 1/2\ A\ T_2.$$

If an expression is written for the acceleration of this waveform, and it is double integrated to solve for the displacement. The displacement boundary conditions can then be used to show that a zero displacement change can be achieved if

$$T_1 = \frac{T}{2} - \frac{2T_2}{3}$$

Thus two parameters are available (B, T_1) to control the velocity and displacement change of the composite waveform. A similar analysis can be done for other bias pulses and classical waveforms. If both the bias pulse and the classical waveform are symmetrical the solution is always to place the classical waveform in the center of the bias pulse.

A bias pulse which has been found to work quite satisfactorily is a cosine bell or Hanning pulse. The advantages of this bias pulse are: The waveform is zero and smooth at the ends, which usually improves the reproduction. The energy of the pulse is concentrated at the lower frequencies with a well-defined upper frequency and a sharp roll-off characteristic (18 dB/oct). This means that the effects of the bias pulse will be seen only at the lower frequencies. The frequency where the effects will start to be seen can be controlled with the duration of the bias pulse.

Least Favorable Response Techniques [17]

The basic assumption of the least favorable response techniques is that the magnitude of the Fourier spectrum has been specified. Since the undamped residual shock spectrum is related to the Fourier modulus, it could also be specified. It is then shown that if the frequency transfer function between the mechanical input and the response of a critical point on the test item (not the transfer function of the shaker system) can be characterized by

$$H(\omega) = |H(\omega)|\ e^{i\theta(\omega)}\ ,$$

then the peak response of the critical point on the structure will be maximized by the input

$$X(\omega) = X_e(\omega)e^{-i\theta(\omega)}\ ,$$

where $X_e(\omega)$ is the specified Fourier modulus, and

$$\ddot{x}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{i\omega t}\ d\omega.$$

where $\ddot{x}(t)$ is the required time history at the input to the test item.

The Fourier transform of the response of the critical point ($Y(\omega)$) is given by

$$Y(\omega) = X(\omega)H(\omega)$$

$$= X_e(\omega)\ |\ H(\omega)\ |\ .$$

The function $Y(\omega)$ is real. This response can be seen to produce a maximum response by thinking of a transient as being composed of a sum of many sinusoids. If the phase angle of all the sinusoids is zero at some point in time all the components will constructively add, and produce the largest possible peak. It is also shown in Appendix A that this response will have a minimum rms duration.

Modern methods make the computation of the above equations relatively easy. It is proposed that the phase angle (Θ) of the frequency transfer function be measured in the laboratory. This function, together with the specified Fourier modulus ($X_e(\omega)$) is then used to compute the test input $\ddot{x}(t)$. Note that the above technique considers the actual test item response, while the shock spectrum approach considers only the response of single degree of freedom systems with known damping. The least favorable response method does assume a linear system with a well-defined critical response.

The method does guarantee that the largest possible peak response will be achieved (hence a guarantee of a conservative test). Shock spectrum techniques cannot make this guarantee for multiple degree of freedom systems. Several examples [17, 18] using the LFR method have indicated peak responses on the order of 1 to 2.5 times a typical peak response to a field event.

This method has been used on one known test series [18] and the results are compared with the shock spectrum methods. As the method requires considerable computations and the reproduction of a complicated computed waveform, digital methods are required for its application.

A spin-off, from the least favorable response techniques, is to assume a unity transfer function (i.e., $H(\omega) = 1$). This means that the peak input will be maximized. If we think of a transient as being a finite amount of energy with a specified frequency distribution (Fourier modulus), the procedure will produce a transient which will transfer this energy to the test item with the largest possible peak input and in a minimum amount of time.

The input to the test item is given by

$$\ddot{x}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_e(\omega) e^{i\omega t}\, d\omega .$$

Note that $X_e(\omega)$ is a real positive function. Therefore, x(t) will be a real even function. As a result, a typical input defined in this manner will look similar to a SHOC pulse. The resulting input is independent of the test item characteristics, and hence eliminates the need to define the transfer function $H(\omega)$. The only required parameter is the Fourier modulus (or the equivalent undamped residual shock spectrum). For example, the least favorable input with a unity transfer function for a transient with the same Fourier modulus as the transient shown in Figure 14 is given as Figure 26. The rms duration of this transient is 64 msec. One test series [18] indicates that this may be a reasonable approach.

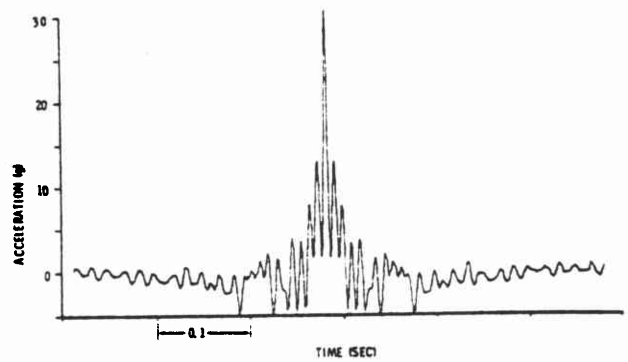


Fig. 26 - A Least Favorable Input with a Unity Transfer Function

ADDITIONAL PARAMETERS (IN ADDITION TO THE SHOCK SPECTRUM) WHICH MAY BE SPECIFIED

Since there are many methods for matching the shock spectrum, it would be desirable to specify additional parameters limiting the class of functions which may be used for a particular test. Several parameters have been suggested and a short discussion of several methods will follow.

Limit the Duration of the Transient

It has been suggested that limits could be placed on the minimum and maximum allowable durations for the transients. It is felt that if the shock spectrum is matched and the duration is comparable, the "damage" should be nearly the same. For complex waveforms, careful attention should be given as to how the duration is defined. The author is not aware of any test program where this specification has actually been used.

Require the Shock Spectrum at Two Different Values of Damping to be Matched

Since damping is the poorest known parameter in many systems, it is felt that if the shock spectrum is matched at two different values of damping (for example, a Q of 5 and 25) the resulting transient should be a reasonable simulation for all values of damping. However, it is a very difficult problem to find a transient which will match a shock spectrum at two values of damping except for a limited class of functions. In fact it is not even clear that a solution always exists. It is true that a solution can exist, for example, a set of shock spectra with different damping values for a known time history. But if those spectra are modified (for example, smoothed, raised in level, or enveloped), it is not clear that a solution will still exist.

Specify the Allowable Ratios of the Peak Shock Response and the Peak Input Level

The attempt is to prevent or encourage the use of oscillatory-type input as opposed to a single-pulse (for example, a half-sine) input. Note that, if the shock spectrum is plotted at a high enough frequency, the maximax shock spectrum reflects the peak input level and this specification is redundant.

This specification would be useful to confirm the value of the maximax spectrum at very high frequencies. TRW has included this requirement in a recent specification which required that the amplification of the response spectrum over the peak-g input would be in the range of 2.5 to 5. This requirement was added to prevent a single pulse or a very long transient from being used.

Specifically Exclude Certain Methods

If the test requester is aware that certain methods are not likely to produce a good simulation, a very effective measure is to simply exclude their use. The limit of this procedure is to exclude all the methods but one. In fact, a knowledgeable requester can specify not only the method, but the parameters needed to define the transient. For example, if decaying sinusoids are specified, then the component frequencies, decay rates, and amplitudes can be listed. In essence, the specification will require a certain time history be used. It should be remembered that if this is done, only those testing laboratories which have implemented the particular method can perform the test. Thought must also be given to requirements for a successful test. One is tempted to place requirements on the fidelity of the time history reproduction. However, most available techniques for reproducing a time history assume a linear system. If the test item is non-linear, the reproduction of a given time history may be poor through no fault of the testing laboratory.

CONCLUSIONS

It is apparent from the preceding discussion that a large variety of methods are now available for matching a shock spectrum on a shaker system. These methods can produce quite different waveforms, and it is not clear that they will all produce equivalent results. It is important that a shock spectrum not be considered as a sufficient specification to define an environment. The "characteristics" of the field waveform should be considered. The personnel who write test specifications must be aware that several procedures could be used to match a specified shock spectrum, and careful consideration should be given as to which procedure will (or will not) produce an acceptable test.

Nomenclature

A - amplitude

a - constant

B - a constant

b - constant, also a frequency

c - constant

E - expected value

f - frequency

g - a basic waveform from which a composite is formed, also an exponential window

i - $\sqrt{-1}$

K - constant

L - the number of basic waveforms which have been added to form a composite waveform

M - slope on a log-log plot in dB/octave

N - number of half cycles, an odd integer

N' - an effective number of cycles at each frequency in a fast sine sweep

Q - quality factor, $\frac{1}{2\zeta}$

R - a constant

S - nondimensional slope on a log-log plot, also auto (power) spectral density

T - a pulse duration

t - time

U - a unit step function

X - Fourier transform of $\ddot{x}$

X_e - a specified Fourier modulus

$\ddot{x}$ - acceleration

$\dot{x}$ - velocity

x - displacement

β - a constant

Γ - a constant

ζ - a decay rate

η - damping coefficient for a shock response spectrum

θ - phase angle

τ - a time delay or shift

ω - circular frequency

Subscripts

c - a velocity and displacement compensating pulse

i - an index

m - an index

Superscripts

* - complex conjugate

REFERENCES

1. G. W. Painter and H. J. Perry, "Simulating Flight Environment Shock on an Electrodynamic Shaker," Shock and Vibration Bulletin, No. 33, Part 3, March 1964, pp. 85-96.

2. J. D. Favour, J. D. LeBrun, and J. P. Young, "Transient Waveform Control of Electromagnetic Test Equipment," Shock and Vibration Bulletin, No. 40, 1969.

3. D. O. Smallwood and A. R. Nord, "Matching Shock Spectra with Sums of Decaying Sinusoids Compensated for Shaker Velocity and Displacement Limitations," Shock and Vibration Bulletin, No. 44, Part 3, pp. 43-56, 1974.

4. D. B. Nelson and P. H. Prasthofer, "A case for Damped Oscillatory Excitation as a Natural Pyrotechnic Shock Simulation," Shock and Vibration Bulletin No. 44, Part 3, pp. 57-71, 1974.

5. J. Carden, T. K. DeClue, and P. A. Koen, "A Vibro-Shock Test System for Testing Large Equipment Items," 44th Shock and Vibration Bulletin, Supplement 1, August 1974, pp. 1-26.

6. R. C. Yang, "Safeguard BMD System - Development of a Waveform Synthesis Technique," Document No. SAF-64, The Ralph M. Parsons Company, 28 August, 1970.

7. R. C. Yang and H. R."Saffell Development of a Waveform Synthesis Technique - A Supplement to Response Spectrum as a Definition of Shock Environment," Shock and Vibration Bulletin, No. 42, Part 2, January 1972, pp. 45-53.

8. D. O. Smallwood and A. F. Witte, "The Use of Shaker-Optimized Periodic Transients in Matching Field Shock Spectra," Shock and Vibration Bulletin, No. 43, Part 1, pp. 139-150.

9. J. D. Crum and R. L. Grant, "Transient Pulse Development," Shock and Vibration Bulletin, No. 41, Part 5, December 1970, pp. 167-176.

10. R. C. Rountree and C. R. Freberg, "Identification of an Optimum Set of Transient Sweep Parameters for Generating Specified Shock Spectra," Shock and Vibration Bulletin, No. 44, Part 3, 1974, pp. 177-192.

11. R. Levy, F. Kozin, and R. B. B. Moorman, "Random Processes for Earthquake Simulation," Journal of the Engineering Mechanics Division, Proceedings of the American Society of Civil Engineers, April 1971, pp. 495-517.

12. R. L. Barnoski and J. R. Maurer, "Transient Characteristics of Single Systems of Modulated Random Noise," Journal of Applied Mechanics, March 1973, pp. 73-77.

13. L. L. Bucciarelli and J. Askinazi, "Pyrotechnic Shock Synthesis Using Nonstationary Broad Band Noise," Journal of Applied Mechanics, June 1973, pp. 429-432.

14. Nien-Chien Tsai, "Spectrum-Compatible Motions for Design Purposes," Journal of the Engineering Mechanics Division, Proceedings of the American Society of Civil Engineers, April 1972, pp. 345-356.

15. D. O. Smallwood, " Methods Used to Match Shock Spectra Using Oscillatory Transients," Institute of Environmental Sciences, 1974 Proceedings, pp. 409-420.

16. R. H. Scanlan and K. Sachs, "Earthquake Time Histories and Response Spectra," Journal of the Engineering Mechanics Division, Proceedings of the American Society of Civil Engineers, Volume 100, No. EM4, August 1974, pp. 635-655.

17. D. O. Smallwood, "A Transient Vibration Test Technique Using Least Favorable Responses," Shock and Vibration Bulletin No. 43, Part 1, pp. 151-164.

18. A. F. Witte and D. O. Smallwood, "A comparison of Shock Spectra and the Least Favorable Response Techniques in a Transient Vibration Test Program," Institute of Environmental Sciences, 1974 Proceedings, pp. 16-29.

19. A. Papoulis, The Fourier Integral and Its Applications, McGraw-Hill, 1962, p. 62.

APPENDIX A
RMS Duration of a Transient

A transient f(t) is defined with a Fourier transform F(ω). The rms duration (D) of the transient is defined as,

$$D^2 = \frac{1}{E}\int_{-\infty}^{\infty} t^2 |f(t)|^2 dt \qquad \text{(A-1)}$$

where

$$E = \int_{-\infty}^{\infty} |f(t)|^2 dt. \qquad \text{(A-2)}$$

E is sometimes referred to as the energy of the pulse. It will be required that E is finite, requiring that f(t) approach zero faster than $1/t^2$, as t approaches both positive and negative infinites. In a general sense, the rms duration of a transient will be a function of the time origin chosen. To avoid this difficultly, it is requried that the time origin be chosen in such a manner as to minimize the rms duration. If some other origin is chosen a time shift (T) can be introduced which will minimize the rms duration,

$$T = \frac{1}{E}\int_{-\infty}^{\infty} t|f(t)|^2 dt. \qquad \text{(A-3)}$$

The rms duration is a measure of the central tendency of a transient. For example, consider a transient of some finite energy composed of all frequencies in equal amounts. An impulse (or delta function) will represent the transient in this class with a minimum duration. A long-duration, low-level random waveform will represent the transient with a maximum duration. The rms duration of several common transients is given in Table A-1.

It can be shown [19] that the rms duration is also given by,

$$D^2 = \frac{1}{2\pi E}\int_{-\infty}^{\infty}\left[\left(\frac{dA}{d\omega}\right)^2 + A^2\left(\frac{d\phi}{d\omega}\right)^2\right]d\omega \qquad \text{(A-4)}$$

where

$$F(\omega) = A(\omega)e^{i\phi(\omega)}.$$

If A(ω) is specified the minimum rms duration is given by

$$\frac{d\phi}{d\omega} = 0$$

or

$$\phi(\omega) = \text{constant}$$

The constant can be zero. Eq. A-4 implies that the rms duration is related to the smoothness of the Fourier spectrum, both the magnitude and the phase. The smoother the Fourier spectrum, the shorter the rms duration.

TABLE A-1

The rms Duration of Some Common Transients

Function	Equation	rms Duration
Square Wave	$f(t) = 1 \quad 0<t<T$ $= 0 \quad$ elsewhere	0.29 T
half sine	$f(t) = \sin\frac{\pi t}{T} \quad 0<t<T$ $= 0 \quad$ elsewhere	0.23 T
terminal peak sawtooth	$f(t) = t/T \quad 0<t<T$ $= 0 \quad$ elsewhere	0.19 T
triangle	$f(t) = 1 - 2\lvert t\rvert/T \quad -\frac{T}{2}<t<\frac{T}{2}$ $= 0 \quad$ elsewhere	0.16 T
haversine	$f(t) = \frac{1}{2}\left(1 + \cos\frac{2\pi t}{T}\right) \quad 0<t<T$ $= 0 \quad$ elsewhere	0.14 T
parabolic cusp	$f(t) = \left(\frac{2}{T}\lvert t\rvert - 1\right)^2 \quad -\frac{T}{2}<t<\frac{T}{2}$ $= 0 \quad$ elsewhere	0.11 T
exponential single side	$f(t) = e^{-at} \quad t>0$ $= 0 \quad t<0$	$\frac{1}{2a}$
exponential	$f(t) = e^{-a\lvert t\rvert}$	$\frac{1}{\sqrt{2}a}$
exponential decaying sinusoid	$f(t) = e^{-a\omega t}\sin\omega t \quad t>0$ $= 0 \quad t<0$ for $a \ll 1$	$\frac{1}{2a\omega}$