CALCULATING TRANSFER FUNCTIONS FROM NORMAL MODES
Revision E

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Variables

F	Excitation frequency
f_r	Natural frequency for mode *r*
N	Total degrees-of-freedom
$H_{ij}(f)$	The steady state displacement at coordinate *i* due to a harmonic force excitation only at coordinate *j*
ξ_r	Damping ratio for mode *r*
ϕ_{ir}	Mass-normalized eigenvector for physical coordinate *i* and mode number *r*
ω	Excitation frequency (rad/sec)
ω_r	Natural frequency (rad/sec) for mode *r*

Receptance

The steady-state displacement at coordinate *i* due to a harmonic force excitation only at coordinate *j* is:

$$H_{ij}(f) = \sum_{r=1}^{N} \left\{ \frac{\phi_{ir}\ \phi_{jr}}{\omega_r^{2}} \frac{1}{\left(1-\rho_r^{2}\right) + \hat{j}\left(2\xi_r\rho_r\right)} \right\} \tag{1}$$

where

$$\rho_r = f / f_r \tag{2}$$

$$\hat{j} = \sqrt{-1} \tag{3}$$

Note that the phase angle is typically represented as the angle by which force leads displacement. In terms of a C++ or Matlab type equation, the phase angle would be

$$\text{Phase} = -\,\text{atan2}(\text{imag}(H), \text{real}(H)) \tag{4}$$

Note that both the phase and the transfer function vary with frequency.

A more formal equation is

$$\text{Phase}(f) = -\arctan\left\{ \frac{\text{imag}\left(H_{ij}(f)\right)}{\text{real}\left(H_{ij}(f)\right)} \right\} \tag{5}$$

Mobility

The steady-state velocity at coordinate *i* due to a harmonic force excitation only at coordinate *j* is

$$H_{ij}(f) = j\omega \sum_{r=1}^{N} \left\{ \frac{\phi_{ir}\ \phi_{jr}}{\omega_r^{\,2}} \frac{1}{\left(1-\rho_r^{\,2}\right) + \hat{j}\ \left(2\xi_r\rho_r\right)} \right\} \tag{6}$$

Accelerance

The steady-state acceleration at coordinate *i* due to a harmonic force excitation only at coordinate *j* is

$$H_{ij}(f) = -\omega^2 \sum_{r=1}^{N} \left\{ \frac{\phi_{ir}\ \phi_{jr}}{\omega_r^{\,2}} \frac{1}{\left(1-\rho_r^{\,2}\right) + \hat{j}\ \left(2\xi_r\rho_r\right)} \right\} \tag{7}$$

Relative Displacement

Consider two translational degrees-of-freedom *i* and *j*. A force is applied at degree-of-freedom *k*.

The steady-state relative displacement transfer function R_{ij} between *i* and *j* due to an applied force at *k* is

$$R_{ij} = H_{ik}(f) - H_{jk}(f)$$

$$= \sum_{r=1}^{N} \left\{ \frac{\phi_{ir}\ \phi_{kr}}{\omega_r^{\,2}} \frac{1}{\left(1-\rho_r^{\,2}\right)+\hat{j}\ \left(2\xi_r\rho_r\right)} \right\} - \sum_{r=1}^{N} \left\{ \frac{\phi_{jr}\ \phi_{kr}}{\omega_r^{\,2}} \frac{1}{\left(1-\rho_r^{\,2}\right)+\hat{j}\ \left(2\xi_r\rho_r\right)} \right\} \tag{8}$$

$$R_{ij} = \sum_{r=1}^{N} \left\{ \frac{\left(\phi_{ir} - \phi_{jr}\right)\phi_{kr}}{\omega_r^{\,2}} \frac{1}{\left(1-\rho_r^{\,2}\right)+\hat{j}\ \left(2\xi_r\rho_r\right)} \right\} \tag{9}$$

The steady-state relative displacement transfer function R_{ij} between *i* and *j* due to an applied force at *k* is

$$R_{ij} = H_{ik}(f) - H_{jk}(f)$$

$$= \sum_{r=1}^{N} \left\{ \frac{\phi_{ir}\ \phi_{kr}}{\omega_r^{\,2}} \frac{1}{\left(1-\rho_r^{\,2}\right)+\hat{j}\ \left(2\xi_r\rho_r\right)} \right\} - \sum_{r=1}^{N} \left\{ \frac{\phi_{jr}\ \phi_{kr}}{\omega_r^{\,2}} \frac{1}{\left(1-\rho_r^{\,2}\right)+\hat{j}\ \left(2\xi_r\rho_r\right)} \right\} \tag{10}$$

$$R_{ij} = \sum_{r=1}^{N} \left\{ \frac{\left(\phi_{ir} - \phi_{jr}\right)\phi_{kr}}{\omega_r^{\,2}} \frac{\left(1-\rho_r^{\,2}\right)}{\left(1-\rho_r^{\,2}\right)^2 + \left(2\xi_r\rho_r\right)^2} \right\}$$

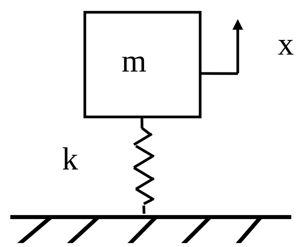
$$- j \sum_{r=1}^{N} \left\{ \frac{\left(\phi_{ir} - \phi_{jr}\right)\phi_{kr}}{\omega_r^{\,2}} \frac{\left(2\xi_r\rho_r\right)}{\left(1-\rho_r^{\,2}\right)^2 + \left(2\xi_r\rho_r\right)^2} \right\} \qquad (11)$$

<u>Reference</u>

1. R. Craig & A. Kurdila, Fundamentals of Structural Dynamics, Second Edition, Wiley, New Jersey, 2006.

APPENDIX A

EXAMPLE 1



Consider the system in Figure A-1. Assign the values in Table A-1.

Table A-1. Parameters		
Variable	Value	Unit
m	1.0	lbf sec^2/in
k	3.946e+05	lbf/in
Damping Ratio	0.05	-

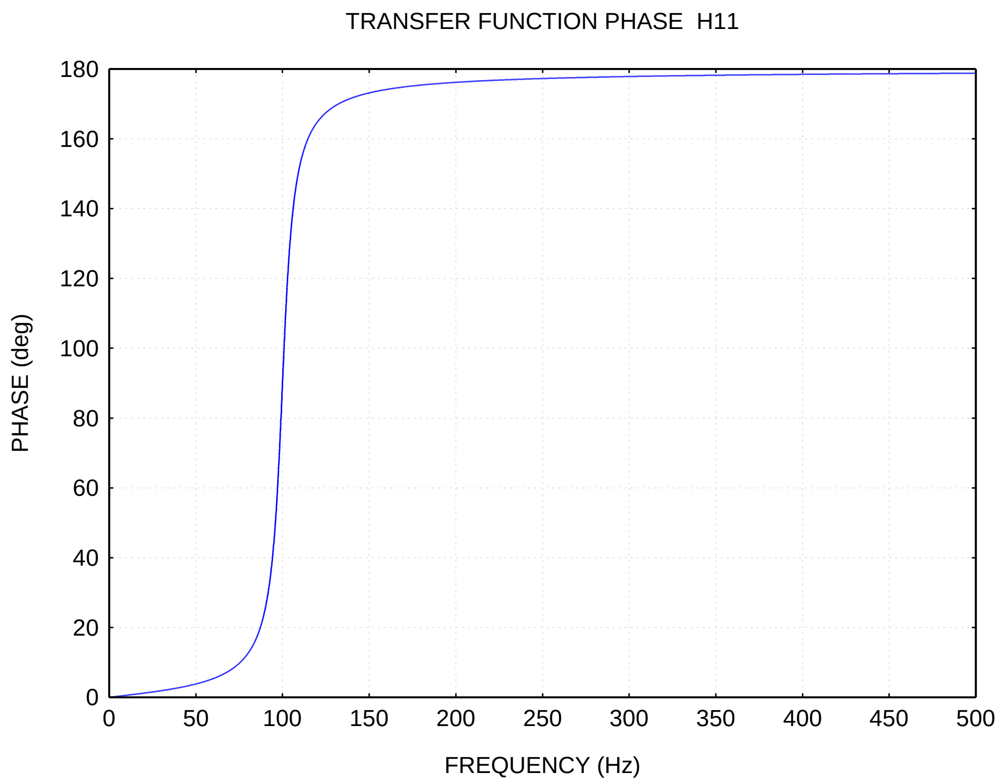


Figure A-1.

This is the phase angle by which the force leads the response.

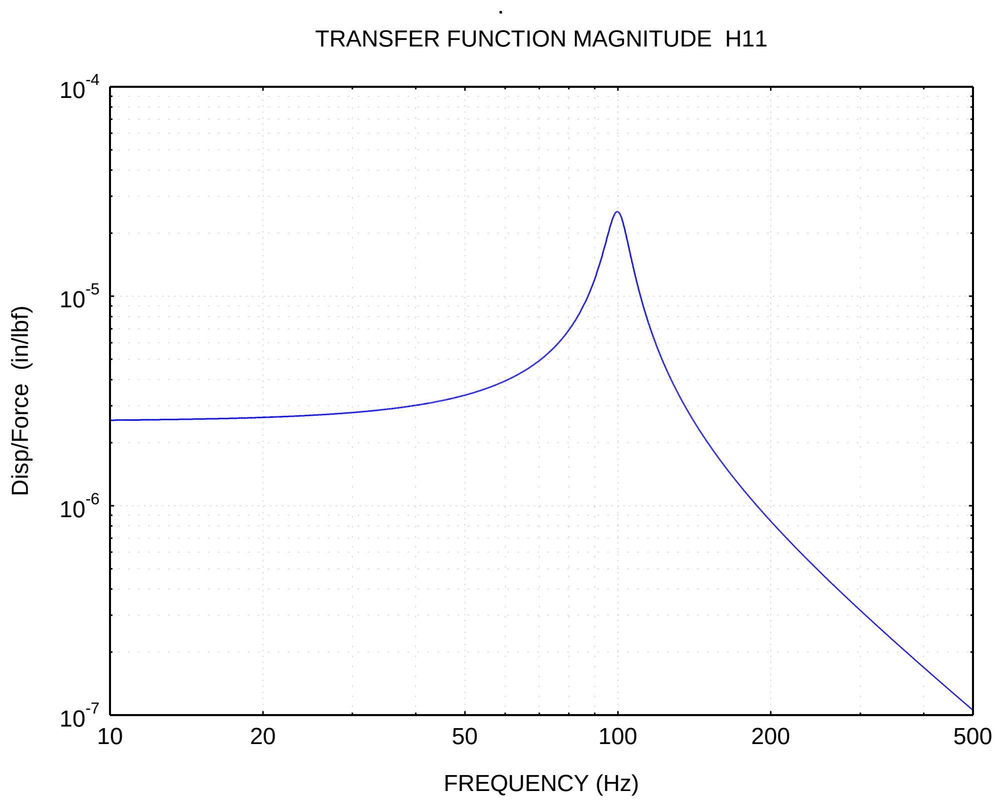

TRANSFER FUNCTION MAGNITUDE H11
10-4
10-5
10-6
10-7
Disp/Force (in/lbf)
10
20
50
100
200
500
FREQUENCY (Hz)

Figure A-2.

APPENDIX B

EXAMPLE 2

Normal Modes Analysis

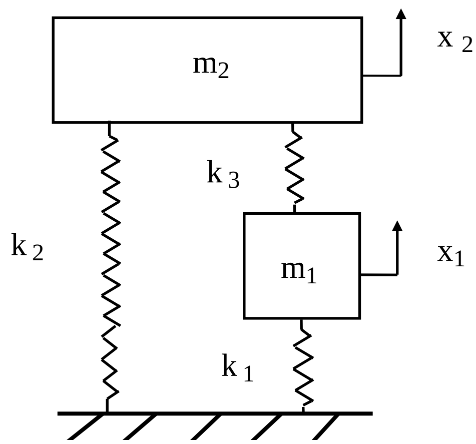


Figure B-1.

Consider the system in Figure B-1. Assign the values in Table B-1.

Table B-1. Parameters

Variable	Value	Unit
m_1	3.0	lbf sec^2/in
m_2	2.0	lbf sec^2/in
k_1	400,000	lbf/in
k_2	300,000	lbf/in
k_3	100,000	lbf/in

Furthermore, assume

1. Each mode has a damping value of 5%.
2. Zero initial conditions

The homogeneous, undamped problem is

$$\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} \ddot{z}_1 \\ \ddot{z}_2 \end{bmatrix} + \begin{bmatrix} 500{,}000 & -100{,}000 \\ -100{,}000 & 400{,}000 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{B-3}$$

The eigenvalue problem is

$$\begin{bmatrix} 500{,}000 - 2\omega^2 & -100{,}000 \\ -100{,}000 & 400{,}000 - \omega^2 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{B-4}$$

The analysis is performed using Matlab script: transfer_from_modes.m

```
>> transfer_from_modes
 transfer_from_modes.m   ver 1.4   June 4, 2010
 by Tom Irvine

 This program calculates a transfer function (displacement/force)
 for each degree-of-freedom in a system based on the mode shapes,
 natural frequencies, and damping ratios.

 Select input method
  1=mass & stiffness matrices
  2=natural frequencies and mass-normalized eigenvectors
 1

 Select output metric
  1=displacement/force
  2=velocity/force
  3=acceleration/force
 1

 Enter the mass matrix name:  mm

mass =

     3     0
     0     2


 Divide mass by 386?
 1=yes 2=no
 2
```

```
 Enter the stiffness matrix name:  kk

stiffness =

      500000     -100000
     -100000      400000


  Natural Frequencies (Hz)
     59.39
      75.9

  Modes Shapes (column format)

QE =

   -0.4792   -0.3220
   -0.3943    0.5869
```

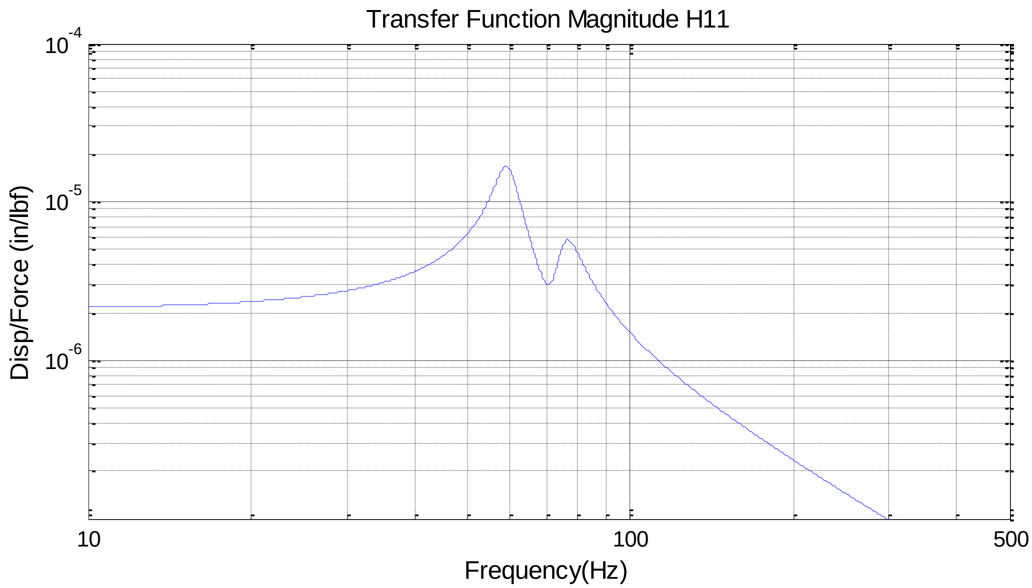


Figure B-2.

The curve is the steady-state displacement at coordinate *1* due to a harmonic force excitation only at coordinate *1*.

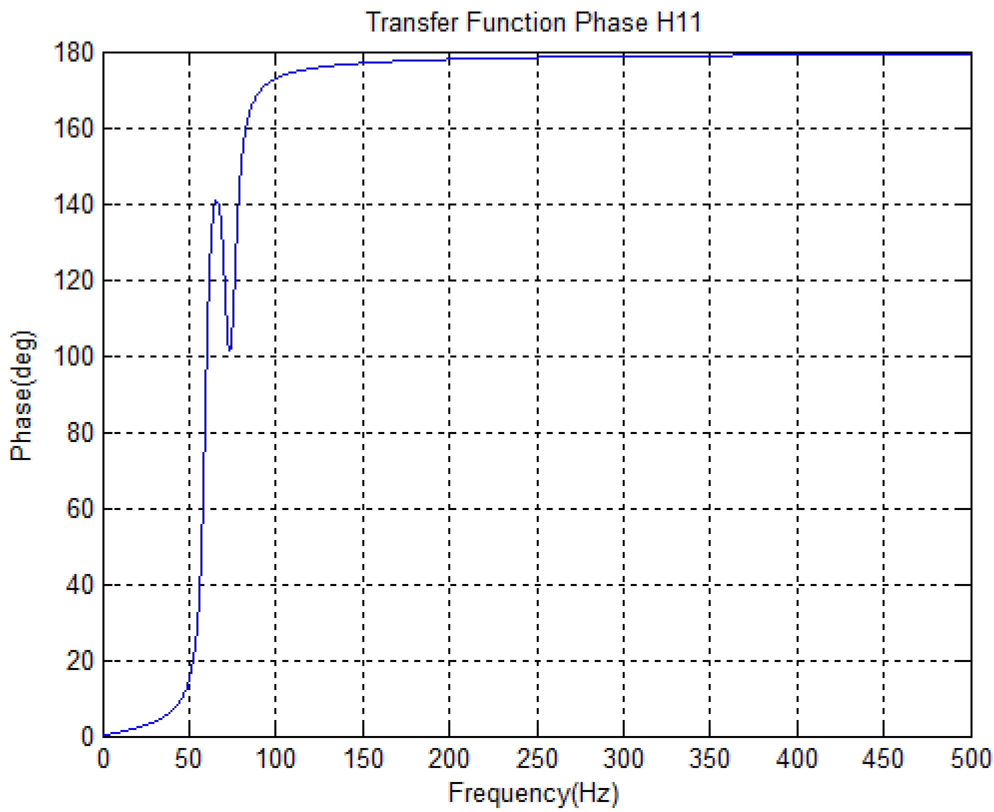

Transfer Function Phase H11
Phase(deg)
Frequency(Hz)
180
160
140
120
100
80
60
40
20
0
0
50
100
150
200
250
300
350
400
450
500

Figure B-3.

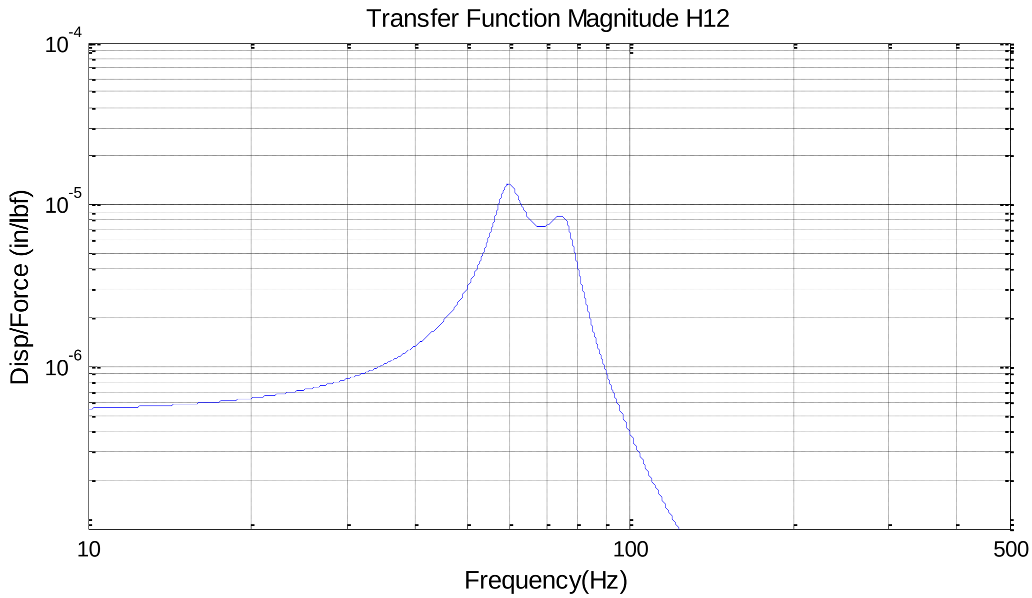


Figure B-4.

The curve is the steady-state displacement at coordinate *1* due to a harmonic force excitation only at coordinate *2.*

Due to reciprocity, it is also the steady-state displacement at coordinate *2* due to a harmonic force excitation only at coordinate *1*.

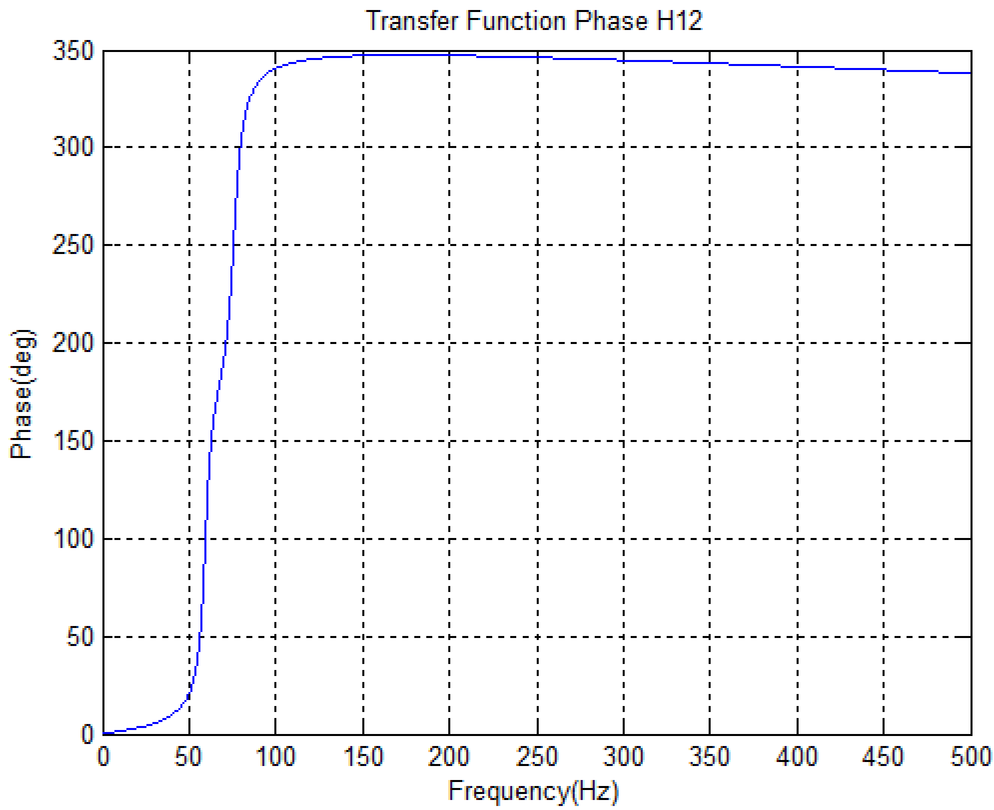

Transfer Function Phase H12
Phase(deg)
350
300
250
200
150
100
50
0
0
50
100
150
200
250
300
350
400
450
500
Frequency(Hz)

Figure B-5.

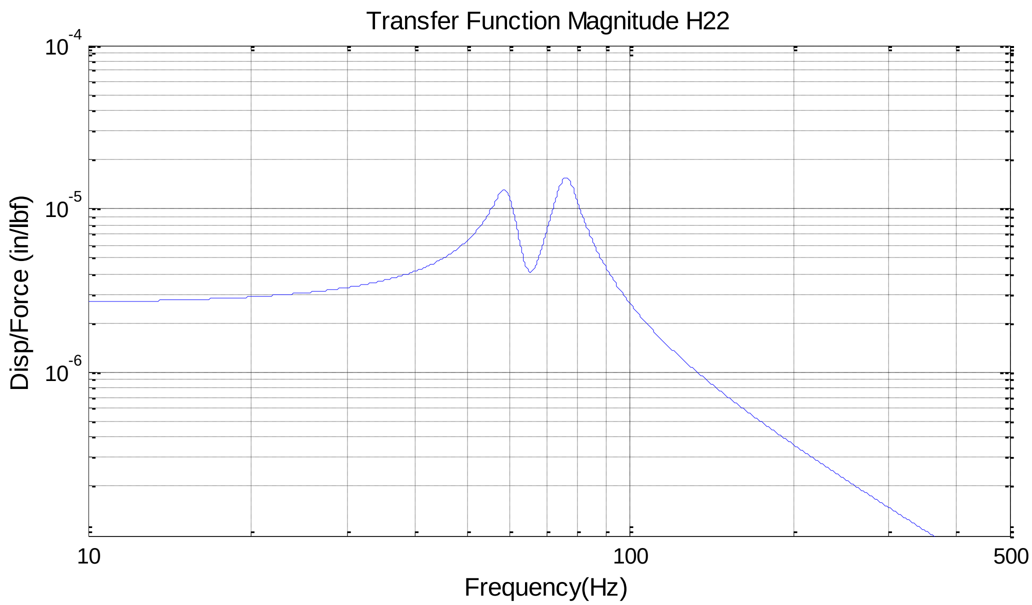


Figure B-6.

The curve is the steady-state displacement at coordinate *2* due to a harmonic force excitation only at coordinate *2.*

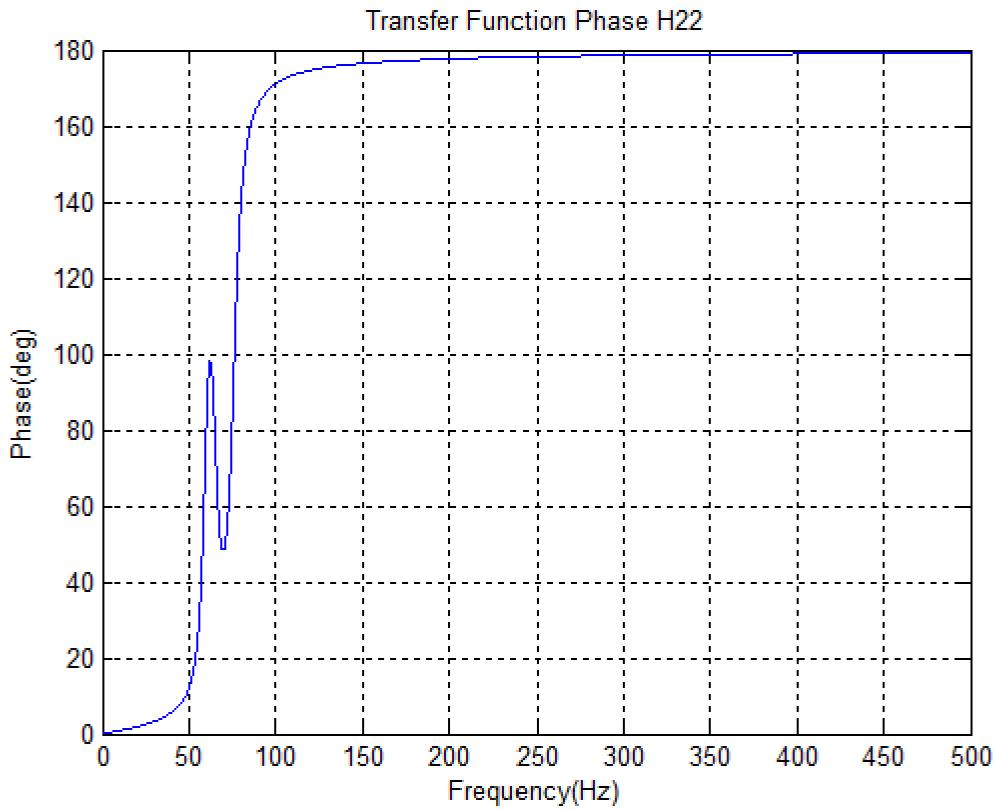

Transfer Function Phase H22
Phase(deg)
Frequency(Hz)
0
20
40
60
80
100
120
140
160
180
0
50
100
150
200
250
300
350
400
450
500

Figure B-7.