TECHNICAL NOTE

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Transverse Vibration of a Simply Supported Frustum of a Right Circular Cone*

REFERENCE: Murphy, J. F., "Transverse Vibration of a Simply-Supported Frustum of a Right Circular Cone," *Journal of Testing and Evaluation,* JTEVA, Vol. 28, No. 5, September 2000, pp. 415–419.

ABSTRACT: Presently. there is no efficient mechanical method in the United States to sort small-diameter logs. This paper explores the problem theoretically using transverse vibration as one such method. The numerical solution to the frequency equation for the transverse vibration of a simply-supported frustum of a right circular cone is found. We refer to this solid as a tapered cylinder with constant taper. The numerical solution is found as a function of cylinder taper, and an approximation to the solution of the frequency equation for slight taper is presented and compared with the numerical solution. This simple yet accurate approximation is most useful to determine the tapered cylinder's flexural stiffness or modulus of elasticity by freely vibrating a simply supported tapered cylinder.

KEYWORDS: transverse vibration, small-diameter logs

Presently there is a need to remove small-diameter trees from forests to restore and sustain forest ecosystem health. To provide an economic incentive and reduce costs associated with small-diameter tree removal in the form of small-diameter logs, a low-cost method to sort the logs for appropriate end use is needed. Modulus of elasticity E is now used in machine stress rating (MSR) of lumber. One possible sorting method for logs is to vibrate the logs transversely to determine flexural stiffness EI and modulus of elasticity. In this report, the vibration of a tapered cylinder is investigated numerically, an approximate formula for the solution of the frequency equation for slight taper is proposed, and the results are compared.

Overview of Solution

We first approximate a tapered cylinder of length L (Fig. 1) by a connected series of constant cross-section cylinders (Fig. 2), each with a slightly smaller radius than the one on its left. Timoshenko et al. [*1*] and Seto [*2*] provide equations of vibration for members with constant cross section. Each sub-cylinder has a particular set of equations reflecting its particular constant cross section. We then match deflection, slope, moment, and shear at the boundaries between sub-cylinders, knowing that the frequency of vibration of each sub-cylinder is the same. This takes the form of a large matrix. Finally, the solution to the frequency equation in the form of a transformed fundamental root is obtained as a function of taper. The frequency equation is the expanded determinant of the matrix, and the numerical solution is obtained when setting the determinant to zero. (Readers not interested in the equations and computational method may proceed to the Results section.)

Method of Solution

To solve the title problem, we first partitioned the tapered cylinder into a stepped cylinder consisting of n equal length cylinders ($\delta = L/n$) of decreasing radii (r_i, see Fig. 2). The start of each tapered sub-cylinder has the larger radius at the left. Each of the stepped sub-cylinders has a constant cross section and therefore constant moment of inertia $I = \pi r_i^4/4$ and constant area $A = \pi r_i^2$. Deflection. slope, moment, and shear are matched across the interface between sub-cylinders. What should the radii of the stepped sub-cylinders be? We look at three possible ways to determine these radii, positioned u_i^* from the right end of the tapered sub-cylinder (Fig. 3):

1. Use the radii at midlength of the tapered sub-cylinder (midlength radii), i.e., $u_{i1}/\delta = 0.5$. Note: u_{i1} is constant for all sub-cylinders, all tapers.
2. Use the radii at u_{i2}/δ the centroid of the tapered sub-cylinder (centroid radii). Note: u_{i2} depends on taper, n, and i (see Appendix).
3. Use the radii at u_{i3}/δ that give equal volume to the stepped sub-cylinder and tapered sub-cylinder (equivolume radii). Note: u_{i3} depends on taper, n, and i (see Appendix).

As calculated, the centroid radii are slightly larger than equivolume radii, which in turn are slightly larger than midlength radii. For clarity, we present the method of solution to the stepped cylinder using the midlength radii. The method is also valid for the centroid radii and equivolume radii. Results and conclusions are valid for all three positions of radii, and for large n they are indistinguishable.

To reduce the number of variables, we use normalized radii at a stepped cylinder. thus

$$\frac{r_i}{r_0} = (b - 1)\left(\frac{i - \frac{1}{2}}{n}\right) + 1$$

where

i = the i^{th} cylinder, $i = 1, \ldots, n$,
n = the number of stepped cylinders,
r_i = the radius of the i^{th} cylinder, constant cross section,

Manuscript received 10/28/99: accepted for publication 5/22/00.

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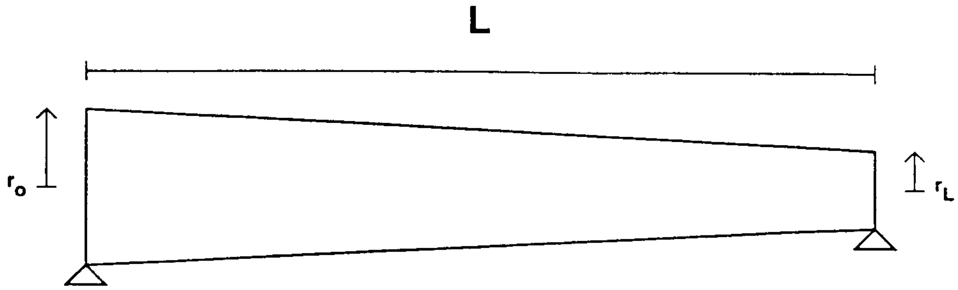

FIG. 1—*Simply supported frustum of a right circular cone.*

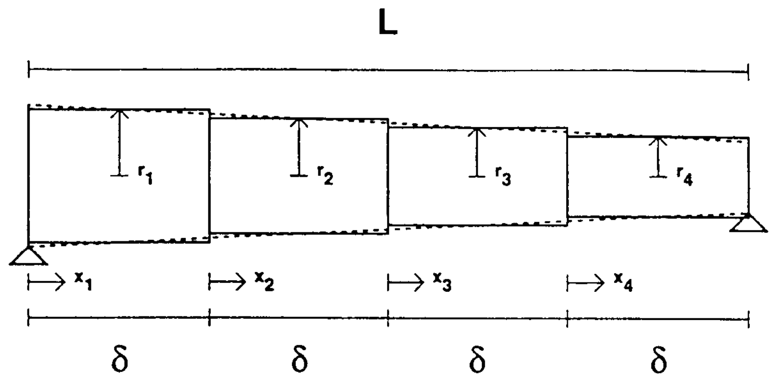

FIG. 2—*Partitioned tapered cylinder into a stepped cylinder* (n = *4*).

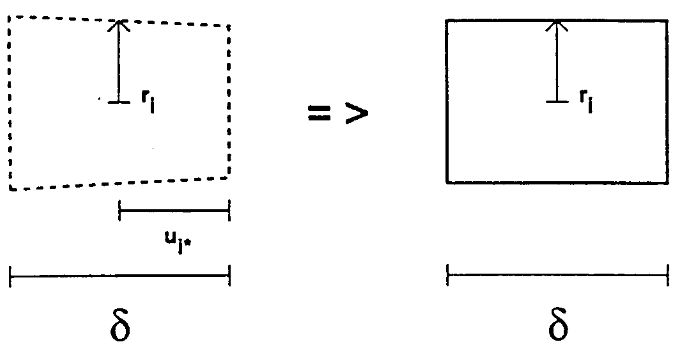

FIG. 3—*Geometric variables for radii of sub-cylinder.*

r_O = the radius at start of tapered cylinder (that is, large end),
b = r_L/r_O, radii ratio of small end to large end (tapered cylinder), a measure of degree of taper,
r_L = the radius of tapered cylinder at L (that is, small end), and
½ = the normalized location for r_i of midlength radii (see previous) (can use u_{i2}/δ or u_{i3}/δ, the centroid or equivolume radii location, respectively).

After setting up the radii of each of the stepped cylinders, we turn to the methodology used by Timoshenko et al. [*1*], Seto [*2*]. and Murphy [*3*], and for brevity we refer the reader to these publications. We assume the large diameter is small in comparison to its length, thereby ignoring any effects of rotary inertia and shearing deformations.[2] Basically, after we divide the tapered cylinder into n divisions, each with its particular origin, then substitute "equivalent" stepped sub-cylinders, we solve the vibration equation for each sub-cylinder and match boundary conditions at the interfaces of the sub-cylinders. Each sub-cylinder has a unique and constant cross section.

The deflection at any location of the i^{th} sub-cylinder varies harmonically with time t as

$$y_i = X_i(A \cos 2\pi ft + B \sin 2\pi ft)$$

where X_i is strictly a function, called a normal function, of distance x_i along the length of the i^{th} sub-cylinder and satisfies a fourth-order ordinary differential equation, $X_i^{iv} - k_i^4 X_i = 0$ [*1*]. The general solution to this differential equation has the form

$$X_i(x_i) = C_{i1} \cos k_i x_i + C_{i2} \sin k_i x_i + C_{i3} \cosh k_i x_i + C_{i4} \sinh k_i x_i$$

where

$$\frac{(2\pi f)^2}{(EI_i/\rho A_i)} = \frac{(2\pi f)^2}{(EI_0/\rho A_0)}\left[(b-1)\left(\frac{i-\frac{1}{2}}{n}\right)+1\right]^{-2}$$

$$= k^4\left[(b-1)\left(\frac{i-\frac{1}{2}}{n}\right)+1\right]^{-2}$$

where
f = cylinder natural frequency,
E = cylinder modulus of elasticity,
I_t = cylinder moment of inertia,
ρ = cylinder mass density, and
A_i = cylinder cross-sectional area.

The previous equation is significant because although each sub-cylinder has its particular value of k_i, the k_i are a constant k times a function of b, i, and n. This allows us to formulate the matrix in terms of b, i, and n and determine how the frequency of vibration is reduced as a function of taper.

[2] Referring to the section in Timenshenko et al. [*1*] on Effects of Rotary Inertia and Shearing Deformations, using a shear coefficient of $k' = 0.9$ for a circular cross section, and assuming a shear modulus equal to 1/16 (0.0625) times the modulus of elasticity for wood, the measured frequency of vibration is reduced by $1 - (2.4d/L)^2$. If the diameter of a wood cylinder is 0.1 times its length, then the effect on frequency is 0.94; if $d/L = 0.0625$, then the effect is 0.98.

We now have n normal functions X_i with $4n$ unknown constants C_{ij}. The necessary $4n$ boundary conditions come from zero moment and zero deflection at the ends of the tapered cylinder and $4(n - 1)$ matching conditions at the n - 1 interfaces (that is, matching deflection, slope, moment, and shear). This $4n \times 4n$ matrix is a banded matrix with an 11-element bandwidth (see Appendix for Matrix Formulation). This set of $4n$ boundary condition equations will have nontrivial solutions only if the determinant of the matrix multiplying the $4n$ coefficients C_{ij} is zero, Thus, the expansion of the $4n \times 4n$ determinant is the frequency equation for free transverse vibrations of an n-stepped cylinder. Roots of the frequency equation. forcing the determinant to vanish. correspond to the natural frequencies. We are interested in the first nonzero root, which corresponds to the fundamental frequency.

The elements of the $4n \times 4n$ matrix consist of trigonometric and hyperbolic terms with arguments kLa_i and multiplied by 1, α_i, α_i^2 , or α_i^3 for deflection. slope. moment. and shear. respectively (see Matrix Formulation in the Appendix). The term α_i is defined as

$$\alpha_i = \frac{1}{n}\left[(b-1)\left(\frac{i-\frac{1}{2}}{n}\right)+1\right]^{-1/2}$$

As in Murphy [3]. we define K_1 as the transformed fundamental root of the frequency equation:

$$K_1 = \left[\frac{(kL)^2}{2\pi}\right]^2$$

$$f^2 = K_1 \frac{EI_0}{L^4 \rho A_0}$$

As can be seen, we define the solution K_1 in reference to a cylinder without taper.

Computational Solution Steps

1. Select a value for n, divide the cylinder into n parts each with length L/n.
2. Select a value for b (that is, degree of taper. r_L/r_O) with $1 \geq b > 0$.
3. Choose a value for kL.
4. Find the determinant of the $4n \times 11$ band matrix by
   (a) using Gaussian elimination with row pivoting to reduce the matrix to an upper banded matrix (can do in-place by adding 2 columns. $4n \times 13$) and
   (b) multiplying the diagonal terms to calculate the determinant.
5. Check the determinant against a very small number and iterate steps 3, 4, and 5 until the determinant is close enough to zero.
6. Calculate $K_1 = [(kL)^2/(2\pi)]^2$; this K_1 is specific for the degree of taper b being investigated.
7. Loop steps 2 to 7, covering degree of taper b from 1 to 0.01.

Results

Numerical Solution

The solid curve in Fig. 4 is K_1 computed as described in the precious section as a function of degree of taper b. This curve is the overlay of the solutions using midlength, centroid, and equivolume radii. with the tapered cylinder divided into 512 equal parts (n = 512), showing no difference resulting from the radius definition used. The number of equations in the matrix is 2048. We get the same numerical solution with 32 divisions. The number of divisions, n, was doubled until the solution remained the same. As b approaches 1 or no taper. K_1 numerically converges to 2.367, which agrees with the analytical solution [*1*] of a simply supported beam-cylinder ($[\pi^2/(2\pi)]^2$).

Numerical Bounds

The uppermost dashed line of Fig. 4 represents K_1 for a no-taper cylinder with radius r_O. The lowermost dashed curve represents K_1 for a no-taper cylinder with radius r_L but using I_O and A_O. This results in K_1 times $(r_L/r_O)^2$ or $K_1 b^2$.

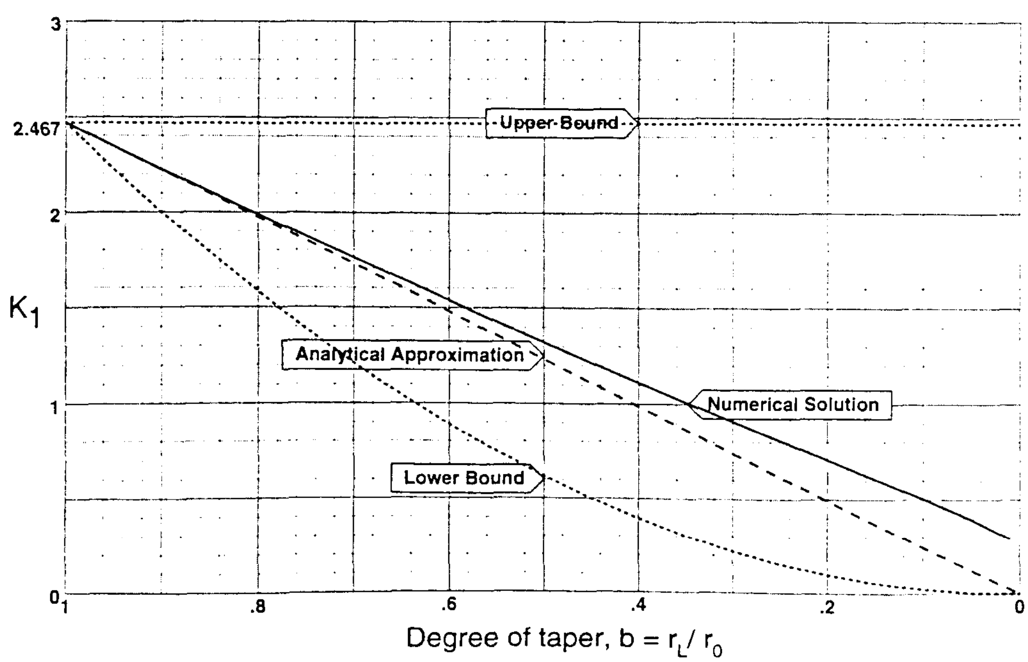

FIG. 4—*Transformed fundamental root of the frequency equation as a function of the degree of taper.*

Approximate Solution

The diagonal dashed line of Fig. 4 is an approximation for the solid line numerical solution for cylinders with small taper, $K_1 = 2.467\, r_L/r_0$. Substituting the approximation for K_1 for small taper results in the following equation for small tapered cylinders:

$$f^2 = 2.467 \frac{EI_t}{L^4 \rho A_t}$$

where I_t and A_t are calculated using the geometric mean square root radius

$$r = \sqrt{r_0 r_L}$$

to give

$$f^2 = 2.467 \frac{E}{L^4 \rho} \frac{r_0 r_L}{4}$$

At $b = 0.85$, the ratio of the approximation to the numerical solution is 0.996, and at $b = 0.80$, the ratio is 0.993.

Conclusions

The numerical solution of the frequency equation of the free transverse vibration of a simply supported frustum of a right circular cone (that is. a tapered cylinder) divided into stepped cylinders has been presented. A simple analytic approximation to the numerical solution for a small taper ($1 \geq b \geq 0.8$) was also developed. This closed-form approximate solution is easier to implement computationally than the numerical solution. From this work, the following conclusions can be stated:

- The numerical solutions using three different radii r_i for the sub-cylinders are indistinguishable.
- The analytic approximation. using the geometric mean square root radius of the two end radii to calculate the moment of inertia and cross-sectional area of the tapered cylinder, results in a slightly unconservative estimate of the tapered cylinder's stiffness EI_t and the modulus of elasticity E.
- Using these equations, transverse vibration can be used as a method to sort small-diameter logs if frequency f, geometry L, r_0, r_L. and mass density ρ are measured.

APPENDIX

Calculation of Centroid Location

The r_j are radii at the ends of the tapered sub-cylinder. The distances u_{i2} locate the centroid of the tapered sub-cylinder.

$$\frac{r_j}{r_0} = b + (1 - b)\frac{n - j}{n} \qquad j = 0, \ldots, n$$

$$q = \left(\frac{r_j/r_0}{r_{j-1}/r_0}\right)^2 \qquad j = 1, \ldots, n$$

$$\frac{u_{i2}}{\delta} = \frac{3 + 2q + q^2}{4 + 4q + 4q^2} \qquad j = 1, \ldots, n, \quad i = j$$

Calculation of Equivolume Location

The r_i are radii that yield the same volume in the stepped sub-cylinder as the tapered sub-cylinder. The distances u_{i3} locate where these radii occur in the tapered sub-cylinder.

$$\left(\frac{r_i}{r_0}\right)^2 = \frac{1}{3r_0^2}(r_j^2 + r_{j-1}^2 + r_j r_{j-1}) \qquad j = 1, \ldots, n$$

$$\frac{u_{i3}}{\delta} = \left(\frac{r_i}{r_0} - \frac{r_j}{r_0}\right) \Bigg/ \left(\frac{r_{j-1}}{r_0} - \frac{r_j}{r_0}\right)$$

Matrix Formulation

The general normal equations for deflection, slope, moment, and shear of the i^{th} sub-cylinder are, respectively ($i = 1, \ldots, n$),

$$X_i = (C_{i1} \cos k_i x + C_{i2} \sin k_i x + C_{i3} \cosh k_i x + C_{i4} \sinh k_i x)$$

$$\frac{dX_i}{dx_i} = k_i(-C_{i1} \sin k_i x + C_{i2} \cos k_i x + C_{i3} \sinh k_i x + C_{i4} \cosh k_i x)$$

$$\frac{d^2X_i}{dx_i^2} = k_i^2(-C_{i1} \cos k_i x - C_{i2} \sin k_i x + C_{i3} \cosh k_i x + C_{i4} \sinh k_i x)$$

$$\frac{d^3X_i}{dx_i^3} = k_i^3(C_{i1} \sin k_i x - C_{i2} \cos k_i x + C_{i3} \sinh k_i x + C_{i4} \cosh k_i x)$$

with the following matching continuity boundary conditions at the end of one cylinder and the start of the next cylinder ($i = 1, \ldots, n - 1$):

$$@x_i = \delta, \qquad @x_{i+1} = 0$$

$$X_i - X_{i+1} = 0$$

$$\frac{dX_i}{dx_i} - \frac{dX_{i+1}}{dx_{i+1}} = 0$$

$$\frac{d^2X_i}{dx_i^2} - \frac{d^2X_{i+1}}{dx_{i+1}^2} = 0$$

$$\frac{d^3X_i}{dx_i^3} - \frac{d^3X_{i+1}}{dx_{i+1}^3} = 0$$

If we substitute

$$\delta = \frac{L}{n}, \qquad k_i\delta = kL\frac{1}{n}\left[(b - 1)\left(\frac{i - 1/2}{n}\right) + 1\right]^{-1/2}$$

$$\text{where} \qquad b = \frac{r_L}{r_0}, \qquad r_0 \geq r_L$$

$$A_i = \pi r_0^2\left[(b - 1)\left(\frac{i - 1/2}{n}\right) + 1\right]^2$$

$$I_i = \frac{\pi r_0^4}{4}\left[(b - 1)\left(\frac{i - 1/2}{n}\right) + 1\right]^4$$

and define

$$\alpha_i = \frac{1}{n}\left[(b-1)\left(\frac{i-\frac{1}{2}}{n}\right)+1\right]^{-\frac{1}{2}}$$

then we can develop the $4n \times 4n$ matrix, which multiplies the $4n$ unknowns C_{ij}.

The elements enforcing zero deflection and zero moment at the left end of sub-cylinder 1 are

$$\begin{matrix} 1 & 0 & 1 & 0 \\ -\alpha_1^2 & 0 & \alpha_1^2 & 0 \end{matrix}$$

The elements for adjacent sub-cylinders. matching deflection, slope, moment, and shear are. respectively ($i = 1, \ldots, n - 1$),

$$\begin{matrix} \cos kL\alpha_i & \sin kL\alpha_i & \cosh kL\alpha_i & \sinh kL\alpha_i & -1 & 0 & -1 & 0 \\ -\alpha_i \sin kL\alpha_i & \alpha_i \cos kL\alpha_i & \alpha_i \sinh kL\alpha_i & \alpha \cosh kL\alpha_i & 0 & -\alpha_{i+1} & 0 & -\alpha_{i+1} \\ -\alpha_i^2 \cos kL\alpha_i & -\alpha_i^2 \sin kL\alpha_i & \alpha_i^2 \cosh kL\alpha_i & \alpha_i^2 \sinh kL\alpha_i & \alpha_{i+1}^2 & 0 & -\alpha_{i+1}^2 & 0 \\ \alpha_i^3 \sin kL\alpha_i & -\alpha_i^3 \cos kL\alpha_i & \alpha_i^3 \sinh kL\alpha_i & \alpha_i^3 \cosh kL\alpha_i & 0 & \alpha_{i+1}^3 & 0 & -\alpha_{i+1}^3 \end{matrix}$$

And finally, the elements enforcing zero deflection and zero moment at the right end of sub-cylinder n are

$$\begin{matrix} \cos kL\alpha_n & \sin kL\alpha_n & \cosh kL\alpha_n & \sinh kL\alpha_n \\ -\alpha_n^2 \cos kL\alpha_n & -\alpha_n^2 \sin kL\alpha_n & \alpha_n^2 \cosh kL\alpha_n & \alpha_n^2 \sinh kL\alpha_n \end{matrix}$$

These elements form a $4n \times 11$ banded matrix. To use row pivoting in place, we added two columns to the upper side of the bandwidth for an $4n \times 13$ banded matrix. We made the matrix elements double precision (64 bits) with $n = 5$ *12.*

References

[*1*] Timoshenko, S., Young, D. H., and Weaver, Jr., W., "*Vibration Problems in Engineering*," Fourth Edition, John Wiley & Sons, New York, 1974.

[*2*] Seto, W. W., "*Schaum's Outline Series, Theory and Problems of Mechanical Vibrations*," McGraw-Hill Book Company, New York, 1964.

[*3*] Murphy, J. F., "Transverse Vibration of a Simply Supported Beam with Symmetric Overhang of Arbitrary Length," *Journal of Testing and Evaluation*, JTEVA, Vol. 25, No. 5, September 1997, pp. 522–524.

Testing Evaluation

A Multidisciplinary Forum for Applied Sciences and Engineering

VOLUME 28, NUMBER 5 SEPTEMBER 2000 CODEN:JTEVA