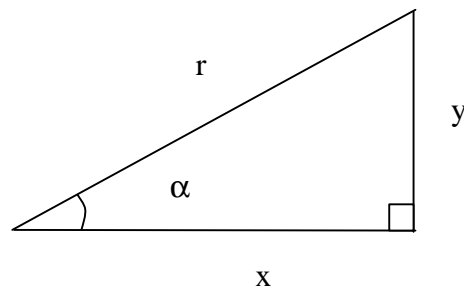


COMPLEX FUNCTIONS AND TRIGONOMETRIC IDENTITIES
Revision D

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Trigonometric Functions of Angle α



$$\sin(\alpha) = \frac{y}{r} \qquad \cos(\alpha) = \frac{x}{r}$$

$$\tan(\alpha) = \frac{y}{x} \qquad \cot(\alpha) = \frac{x}{y}$$

$$\sec(\alpha) = \frac{r}{x} \qquad \csc(\alpha) = \frac{r}{y}$$

(1)

Trigonometric Expansion

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \qquad (2)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \qquad (3)$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots \quad (4)$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots \quad (5)$$

Exponential Expansion

$$\exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \quad (6)$$

Trigonometric Identities

$$\cos^2(\alpha) = \frac{1}{2} + \frac{1}{2}\cos(2\alpha) \quad (7)$$

$$\sin^2(\alpha) = \frac{1}{2} - \frac{1}{2}\cos(2\alpha) \quad (8)$$

$$\cos(\alpha)\cos(\beta) = \frac{1}{2}\cos(\alpha + \beta) + \frac{1}{2}\cos(\alpha - \beta) \quad (9)$$

$$\sin(\alpha)\sin(\beta) = -\frac{1}{2}\cos(\alpha + \beta) + \frac{1}{2}\cos(\alpha - \beta) \quad (10)$$

$$\cos(\alpha)\sin(\beta) = \frac{1}{2}\sin(\alpha + \beta) - \frac{1}{2}\sin(\alpha - \beta) \quad (11)$$

$$\sin(\alpha)\cos(\beta) = \frac{1}{2}\sin(\alpha + \beta) + \frac{1}{2}\sin(\alpha - \beta) \quad (12)$$

$$\sin^2(\alpha) + \cos^2(\alpha) = 1 \quad (13)$$

$$\sec^2(\alpha) - \tan^2(\alpha) = 1 \quad (14)$$

$$\csc^2(\alpha) - \cot^2(\alpha) = 1 \quad (15)$$

$$\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha) \quad (16)$$

$$\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha) \quad (17)$$

$$\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta) \quad (18)$$

$$\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta) \quad (19)$$

$$\tan(\alpha \pm \beta) = \frac{\tan(\alpha) \pm \tan(\beta)}{1 \mp \tan(\alpha)\tan(\beta)} \quad (20)$$

$$A \sin(\alpha t) + B \cos(\alpha t) = \sqrt{A^2 + B^2} [\sin(\alpha t + \theta)]$$

$$\text{where } \theta = \arctan\left(\frac{B}{A}\right) \quad (21)$$

Euler's Equation

$$\exp(\pm j\alpha) = \cos(\alpha) \pm j \sin(\alpha) \quad (22)$$

$$\sin(\alpha) = \frac{\exp(j\alpha) - \exp(-j\alpha)}{2j} \quad (23)$$

$$\cos(\alpha) = \frac{\exp(j\alpha) + \exp(-j\alpha)}{2} \quad (24)$$

Hyperbolic Functions

$$\sinh(\alpha) = \frac{\exp(\alpha) - \exp(-\alpha)}{2} \quad (25)$$

$$\cosh(\alpha) = \frac{\exp(\alpha) + \exp(-\alpha)}{2} \quad (26)$$

Derivatives

$$\frac{d}{dx}(\sin u) = \cos u \frac{du}{dx} \quad (27)$$

$$\frac{d}{dx}(\cos u) = -\sin u \frac{du}{dx} \quad (28)$$

$$\frac{d}{dx}(\tan u) = \sec^2 u \frac{du}{dx} \quad (29)$$

$$\frac{d}{dx}(\cot u) = -\csc u \frac{du}{dx} \quad (30)$$

$$\frac{d}{dx}(\sec u) = \tan u \sec u \frac{du}{dx} \quad (31)$$

$$\frac{d}{dx}(\csc u) = -\csc u \cot u \frac{du}{dx} \quad (32)$$

$$\frac{d}{dx}(\sinh u) = \cosh u \frac{du}{dx} \quad (33)$$

$$\frac{d}{dx}(\cosh u) = \sinh u \frac{du}{dx} \quad (34)$$

$$\frac{d}{dx}(\tanh u) = \frac{1}{\cosh^2 u} \frac{du}{dx} \quad (35)$$

Natural Logarithm of a Complex Number

$$\ln(a + jb) = \ln\left[\sqrt{a^2 + b^2} \exp(j\theta)\right], \quad \theta = \arctan\left(\frac{b}{a}\right) \quad (36)$$

$$\ln(a + jb) = \ln\left[\sqrt{a^2 + b^2}\right] + \ln[\exp(j\theta)] \quad (37)$$

$$\ln(a + jb) = \ln\left[\sqrt{a^2 + b^2}\right] + j\theta \quad (38)$$

$$\ln(a + jb) = \ln\left[\sqrt{a^2 + b^2}\right] + j \arctan\left(\frac{b}{a}\right) \quad (39)$$

APPENDIX A

The Square Root of a Complex Number.

Consider

$$x^2 = a + jb \quad (A-1)$$

Thus

$$x = \pm\sqrt{a + jb} \quad (A-2)$$

where a and b are real coefficients.

Solve for x.

Let

$$x_1 = (c + jd) \quad (A-3a)$$

$$x_2 = -(c + jd) \quad (A-3b)$$

where c and d are real coefficients.

Substitute equation (A-3a) into (A-1).

$$(c + jd)^2 = (a + jb) \quad (A-4)$$

$$(c + jd)(c + jd) = (a + jb) \quad (A-5)$$

$$c^2 - d^2 + j(2cd) = a + jb \quad (A-6)$$

Equation (A-6) implies two equations. The first is

$$c^2 - d^2 = a \tag{A-7}$$

The second implied equation is

$$2cd = b \tag{A-8}$$

Solve for d using equation (A-8).

$$d = \frac{b}{2c} \tag{A-9}$$

Substitute equation (A-9) into (A-8).

$$c^2 - \left(\frac{b}{2c}\right)^2 = a \tag{A-10}$$

$$c^2 - a - \left(\frac{b}{2c}\right)^2 = 0 \tag{A-11}$$

Multiply through by $4c^2$.

$$4c^4 - 4ac^2 - b^2 = 0 \tag{A-12}$$

Apply the quadratic formula.

$$c^2 = \frac{4a \pm \sqrt{16a^2 + 16b^2}}{8} \tag{A-13}$$

$$c^2 = \frac{4a \pm 4\sqrt{a^2 + b^2}}{8} \tag{A-14}$$

$$c^2 = \frac{a \pm \sqrt{a^2 + b^2}}{2} \tag{A-15}$$

$$c = \sqrt{\frac{a \pm \sqrt{a^2 + b^2}}{2}} \quad (\text{A-16})$$

Require c to be real.

$$c = \sqrt{\frac{a + \sqrt{a^2 + b^2}}{2}} \quad (\text{A-17})$$

Substitute equation (A-17) into (A-9).

$$d = \left\{ \frac{b}{2 \left[\sqrt{\frac{a + \sqrt{a^2 + b^2}}{2}} \right]} \right\} \quad (\text{A-18})$$

$$d = \left\{ \frac{b}{\sqrt{\frac{4(a + \sqrt{a^2 + b^2})}{2}}} \right\} \quad (\text{A-19})$$

$$d = \left\{ \frac{b}{\sqrt{2(a + \sqrt{a^2 + b^2})}} \right\} \quad (\text{A-20})$$

Substitute equations (A-20) and (A-17) into (A-3a).

$$x_1 = \left\{ \sqrt{\frac{a + \sqrt{a^2 + b^2}}{2}} \right\} + j \left\{ \frac{b}{\sqrt{2(a + \sqrt{a^2 + b^2})}} \right\} \quad (\text{A-21a})$$

Substitute equations (A-20) and (A-17) into (A-3b).

$$x_2 = - \left\{ \sqrt{\frac{a + \sqrt{a^2 + b^2}}{2}} \right\} - j \left\{ \frac{b}{\sqrt{2(a + \sqrt{a^2 + b^2})}} \right\} \quad (\text{A-21b})$$

Note that equations (A-21a) and (A-21b) cannot be used for the special case:

$$a < 0 \text{ and } b = 0.$$

For this special case, the roots are

$$x = \pm j\sqrt{a} \quad (\text{A-21c})$$

Example

$$x^2 = 2 + j7 \quad (\text{A-22})$$

$$x = \pm \sqrt{2 + j7} \quad (\text{A-23})$$

Solve for x. Use equation (A-21a).

$$a = 2 \quad (\text{A-24})$$

$$b = 7 \quad (\text{A-25})$$

$$x_1 = 2.154 + j1.625 \quad (\text{A-26})$$

$$x_2 = -2.154 - j1.625 \quad (\text{A-27})$$

APPENDIX B

Arbitrary Root of a Complex Number

Let

$$x^n = [a + jb] \quad (\text{B-1a})$$

$$x = [a + jb]^{1/n} \quad (\text{B-1b})$$

The coefficients a and b are real numbers. The denominator of the exponent n is also real.

Take the natural logarithm.

$$\ln x = \ln \left\{ [a + jb]^{1/n} \right\} \quad (\text{B-2})$$

$$\ln x = \frac{1}{n} \ln [a + jb] \quad (\text{B-3})$$

$$\ln x = \frac{1}{n} \ln \left[\sqrt{a^2 + b^2} \exp \left(j \arctan \frac{b}{a} \right) \right] \quad (\text{B-4})$$

$$\ln x = \frac{1}{n} \ln \left[\sqrt{a^2 + b^2} \right] + \frac{1}{n} \ln \left[\exp \left(j \arctan \frac{b}{a} \right) \right] \quad (\text{B-5})$$

$$\ln x = \ln \left[\left(a^2 + b^2 \right)^{\frac{1}{2n}} \right] + j \frac{1}{n} \arctan \frac{b}{a} \quad (\text{B-6})$$

$$\exp \{ \ln x \} = \exp \left\{ \ln \left[\left(a^2 + b^2 \right)^{\frac{1}{2n}} \right] + j \frac{1}{n} \arctan \frac{b}{a} \right\} \quad (\text{B-7})$$

$$x = \exp \left\{ \ln \left[\left(a^2 + b^2 \right)^{\frac{1}{2n}} \right] \right\} \exp \left\{ j \frac{1}{n} \arctan \frac{b}{a} \right\} \quad (\text{B-8})$$

$$x = \left(a^2 + b^2 \right)^{\frac{1}{2n}} \exp \left\{ j \frac{1}{n} \arctan \frac{b}{a} \right\} \quad (\text{B-9})$$

$$x = \left(a^2 + b^2 \right)^{\frac{1}{2n}} \left\{ \cos \left(\frac{1}{n} \arctan \frac{b}{a} \right) + j \sin \left(\frac{1}{n} \arctan \frac{b}{a} \right) \right\} \quad (\text{B-10})$$

Note that equation (B-10) could be used for the special case of a square root.

APPENDIX C

Cube Root of a Complex Number

Consider

$$x^3 = a + jb \quad (\text{C-1})$$

$$x = [a + jb]^{1/3} \quad (\text{C-2})$$

Equation (C-1) has three roots. The method in Appendix B yields the following formula for one of the cube roots.

$$x_1 = \left(a^2 + b^2 \right)^{\frac{1}{6}} \left\{ \cos \left(\frac{1}{3} \arctan \frac{b}{a} \right) + j \sin \left(\frac{1}{3} \arctan \frac{b}{a} \right) \right\} \quad (\text{C-3})$$

Rearrange equation (C-1).

$$x^3 - a - jb = 0 \quad (\text{C-4})$$

Devise an equation for finding the other two roots.

$$x^3 - a - jb = (x - x_1)(x - x_2)(x - x_3) \quad (\text{C-5})$$

Expand the right-hand-side.

$$x^3 - a - jb = [x^2 - (x_1 + x_2)x + x_1x_2](x - x_3) \quad (\text{C-6})$$

$$x^3 - a - jb = [x^2 - (x_1 + x_2)x + x_1x_2](x) + [x^2 - (x_1 + x_2)x + x_1x_2](-x_3) \quad (\text{C-7})$$

$$x^3 - a - jb = [x^3 - (x_1 + x_2)x^2 + x_1x_2x] - [x_3x^2 - x_3(x_1 + x_2)x + x_1x_2x_3] \quad (\text{C-8})$$

$$x^3 - a - jb = [x^3 - (x_1 + x_2)x^2 + x_1x_2x] + [-x_3x^2 + x_3(x_1 + x_2)x - x_1x_2x_3] \quad (\text{C-9})$$

$$x^3 - a - jb = [x^3 - (x_1 + x_2 + x_3)x^2 + (x_1x_2 + x_1x_3 + x_2x_3)x - x_1x_2x_3] \quad (\text{C-10})$$

Equation (C-10) implies three separate equations.

$$(x_1 + x_2 + x_3) = 0 \quad (\text{C-11})$$

$$(x_1x_2 + x_1x_3 + x_2x_3) = 0 \quad (\text{C-12})$$

$$-x_1x_2x_3 = -a - jb \quad (\text{C-13})$$

Continue with equation (C-11).

$$x_2 = -x_1 - x_3 \quad (\text{C-14})$$

Substitute equation (C-14) into (C-12).

$$(x_1(-x_1 - x_3) + x_1x_3 + (-x_1 - x_3)x_3) = 0 \quad (\text{C-15})$$

$$-x_1^2 - x_1x_3 + x_1x_3 - x_1x_3 - x_3^2 = 0 \quad (\text{C-16})$$

$$-x_1^2 - x_1x_3 - x_3^2 = 0 \quad (\text{C-17})$$

$$x_3^2 + x_1x_3 + x_1^2 = 0 \quad (\text{C-18})$$

Use the quadratic formula.

$$x_3 = \frac{-x_1 \pm \sqrt{x_1^2 - 4x_1^2}}{2} \quad (\text{C-19})$$

$$x_3 = \frac{-x_1 \pm \sqrt{-3x_1^2}}{2} \quad (\text{C-20})$$

$$x_3 = \frac{-x_1 \pm x_1\sqrt{-3}}{2} \quad (\text{C-21})$$

$$x_3 = x_1 \left[\frac{-1 \pm j\sqrt{3}}{2} \right] \quad (\text{C-22})$$

Choose

$$x_3 = x_1 \left[\frac{-1 - j\sqrt{3}}{2} \right] \quad (\text{C-23})$$

Recall equation (C-14).

$$x_2 = -x_1 - x_3 \quad (\text{C-24})$$

$$x_2 = -x_1 - x_1 \left[\frac{-1 - j\sqrt{3}}{2} \right] \quad (\text{C-25})$$

$$x_2 = x_1 \left\{ -1 - \left[\frac{-1 - j\sqrt{3}}{2} \right] \right\} \quad (\text{C-26})$$

$$x_2 = x_1 \left\{ -1 + \left[\frac{1 + j\sqrt{3}}{2} \right] \right\} \quad (\text{C-27})$$

$$x_2 = x_1 \left\{ -\frac{2}{2} + \frac{1 + j\sqrt{3}}{2} \right\} \quad (\text{C-28})$$

$$x_2 = x_1 \left[\frac{-1 + j\sqrt{3}}{2} \right] \quad (\text{C-29})$$

The roots x_2 and x_3 thus form a complex conjugate pair.

Summarize the roots.

$$x_1 = \left(a^2 + b^2 \right)^{\frac{1}{6}} \left\{ \cos \left(\frac{1}{3} \arctan \frac{b}{a} \right) + j \sin \left(\frac{1}{3} \arctan \frac{b}{a} \right) \right\} \quad (\text{C-30})$$

$$x_2 = x_1 \left[\frac{-1 + j\sqrt{3}}{2} \right] \quad (\text{C-31})$$

$$x_3 = x_1 \left[\frac{-1 - j\sqrt{3}}{2} \right] \quad (\text{C-32})$$

Example

Solve for x .

$$x^3 = 2 + j7 \quad (\text{C-33})$$

$$x = [2 + j7]^{1/3} \quad (\text{C-34})$$

$$a = 2 \quad (\text{C-35})$$

$$b = 7 \quad (\text{C-36})$$

$$n = 3 \quad (\text{C-37})$$

There are three roots. The first root is

$$x_1 = \left(a^2 + b^2 \right)^{\frac{1}{6}} \left\{ \cos \left(\frac{1}{3} \arctan \frac{b}{a} \right) + j \sin \left(\frac{1}{3} \arctan \frac{b}{a} \right) \right\} \quad (\text{C-38})$$

$$x_1 = 1.938[0.909 + j418] \quad (\text{C-39})$$

$$x_1 = 1.761 + j 0.809 \quad (\text{C-40})$$

The second root is a coordinate transformation of the first root.

$$x_2 = x_1 \left[\frac{-1 + j\sqrt{3}}{2} \right] \quad (\text{C-41})$$

$$x_2 = [1.762 + j 0.809] \left[\frac{-1 + j\sqrt{3}}{2} \right] \quad (\text{C-42})$$

$$x_2 = -1.581 + j1.120 \quad (\text{C-43})$$

$$x_3 = x_1 \left[\frac{-1 - j\sqrt{3}}{2} \right] \quad (\text{C-44})$$

$$x_3 = [1.761 + j 0.809] \left[\frac{-1 - j\sqrt{3}}{2} \right] \quad (\text{C-45})$$

$$x_3 = -0.180 - j1.930 \quad (\text{C-46})$$

In summary, the cube roots of $(2 + j 7)$ are

$$x_1 = 1.761 + j 0.809 \quad (\text{C-47})$$

$$x_2 = -1.581 + j1.120 \quad (\text{C-48})$$

$$x_3 = -0.180 - j1.930 \quad (\text{C-49})$$

APPENDIX D

Derivation of the Quadratic Formula

$$ax^2 + bx + c = 0 \quad (\text{D-1})$$

$$x^2 + (b/a)x + (c/a) = 0 \quad (\text{D-2})$$

$$\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a} = 0 \quad (\text{D-3})$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a} \quad (\text{D-4})$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} \quad (\text{D-5})$$

$$\left(x + \frac{b}{2a}\right) = \pm \sqrt{\frac{b^2 - 4ac}{4a^2}} \quad (\text{D-6})$$

$$\left(x + \frac{b}{2a}\right) = \frac{\pm \sqrt{b^2 - 4ac}}{2a} \quad (\text{D-7})$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \quad (\text{D-8})$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad (\text{D-9})$$