# TWO-DEGREE-OF-FREEDOM SYSTEM SUBJECTED TO A HALF-SINE PULSE FORCE
Revision A

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Two-degree-of-freedom System

The method of generalized coordinates is demonstrated by an example. Consider the system in Figure 1.

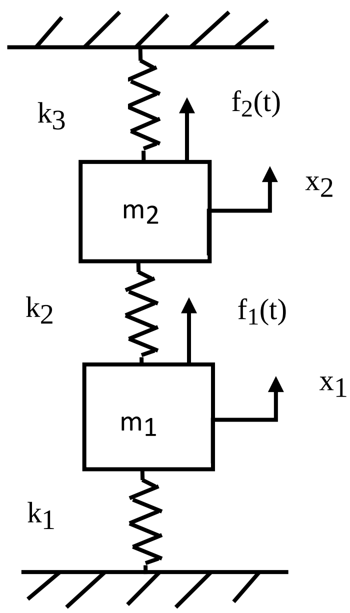


Figure 1.

A free-body diagram of mass 1 is given in Figure 2. A free-body diagram of mass 2 is given in Figure 3.

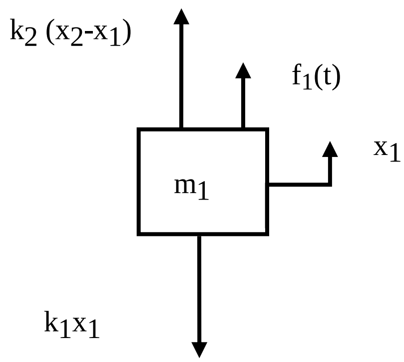


Figure 2.

Determine the equation of motion for mass 1.

$$\sum F = m_1 \ddot{x}_1 \tag{1}$$

$$m_1\ddot{x}_1 = f_1(t) + k_2(x_2 - x_1) - k_1x_1 \tag{2}$$

$$m_1\ddot{x}_1 - k_2(x_2 - x_1) + k_1x_1 = f_1(t) \tag{3}$$

$$m_1\ddot{x}_1 + k_2(-x_2 + x_1) + k_1x_1 = f_1(t) \tag{4}$$

$$m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 = f_1(t) \tag{5}$$

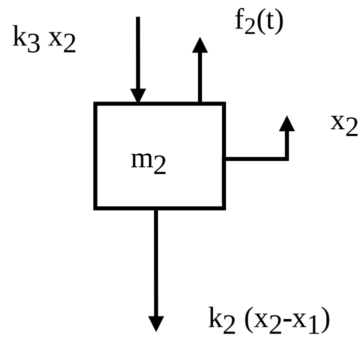


Figure 3.

Derive the equation of motion for mass 2.

$$\sum F = m_2\, \ddot{x}_2 \tag{6}$$

$$m_2\ddot{x}_2 = f_2(t) - k_2(x_2 - x_1) - k_3 x_2 \tag{7}$$

$$m_2\ddot{x}_2 + k_2(x_2 - x_1) + k_3 x_2 = f_2(t) \tag{8}$$

$$m_2\ddot{x}_2 + (k_2 + k_3)x_2 - k_2 x_1 = f_2(t) \tag{9}$$

Assemble the equations in matrix form.

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} \tag{10}$$

Decoupling

Equation (10) is coupled via the stiffness matrix. An intermediate goal is to decouple the equation.

Simplify,

$$M\ddot{\bar{x}} + K\bar{x} = \bar{F} \tag{11}$$

where

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \tag{12}$$

$$K = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \tag{13}$$

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{14}$$

$$\bar{F} = \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} \tag{15}$$

Consider the homogeneous form of equation (11).

$$M\ddot{\bar{x}} + K\bar{x} = \bar{0} \tag{16}$$

Seek a solution of the form

$$\bar{x} = \bar{q}\exp(j\omega t) \tag{17}$$

The q vector is the generalized coordinate vector.

Note that

$$\dot{\bar{x}} = j\omega\bar{q}\exp(j\omega t) \tag{18}$$

$$\ddot{\bar{x}} = -\omega^2\,\bar{q}\exp(j\omega t) \tag{19}$$

Substitute equations (17) through (19) into equation (16).

$$-\omega^2 M \bar{q} \exp(j\omega t) + K \bar{q} \exp(j\omega t) = \bar{0} \tag{20}$$

$$\left\{-\omega^2 M \bar{q} + K \bar{q}\right\} \exp(j\omega t) = \bar{0} \tag{21}$$

$$-\omega^2 M \bar{q} + K \bar{q} = \bar{0} \tag{22}$$

$$\left\{-\omega^2 M + K\right\} \bar{q} = \bar{0} \tag{23}$$

$$\left\{K - \omega^2 M\right\} \bar{q} = \bar{0} \tag{24}$$

Equation (24) is an example of a generalized eigenvalue problem. The eigenvalues can be found by setting the determinant equal to zero.

$$\det\left\{K - \omega^2 M\right\} = \bar{0} \tag{25}$$

$$\det\left\{\begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} - \omega^2 \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\right\} = 0 \tag{26}$$

$$\det\left\{\begin{bmatrix} (k_1 + k_2) - \omega^2 m_1 & -k_2 \\ -k_2 & (k_2 + k_3) - \omega^2 m_2 \end{bmatrix}\right\} = 0 \tag{27}$$

$$\left[(k_1 + k_2) - \omega^2 m_1\right]\left[(k_2 + k_3) - \omega^2 m_2\right] - k_2^2 = 0 \tag{28}$$

$$(k_1+k_2)(k_2+k_3)-\omega^2 m_1(k_2+k_3)-\omega^2 m_2(k_1+k_2)+\omega^4 m_1 m_2-k_2^2=0 \tag{29}$$

$$m_1 m_2\,\omega^4+[-m_1(k_2+k_3)-m_2(k_1+k_2)]\omega^2+k_1 k_3+(k_1+k_3)k_2+k_2^2-k_2^2=0 \tag{30}$$

$$m_1 m_2\,\omega^4+[-m_1(k_2+k_3)-m_2(k_1+k_2)]\omega^2+k_1 k_3+k_1 k_2+k_2 k_3=0 \tag{31}$$

The eigenvalues are the roots of the polynomial.

$$\omega_1^2=\frac{-b-\sqrt{b^2-4ac}}{2a} \tag{32}$$

$$\omega_2^2=\frac{-b+\sqrt{b^2-4ac}}{2a} \tag{33}$$

where

$$a=m_1 m_2 \tag{34}$$

$$b=[-m_1(k_2+k_3)-m_2(k_1+k_2)] \tag{35}$$

$$c=k_1 k_2+k_1 k_3+k_2 k_3 \tag{36}$$

The eigenvectors are found via the following equations.

$$\{K-\omega_1^2\,M\}\bar{q}_1=\bar{0} \tag{37}$$

$$\{K-\omega_2^2\,M\}\bar{q}_2=\bar{0} \tag{38}$$

where

$$\bar{q}_1 = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{39}$$

$$\bar{q}_2 = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \tag{40}$$

An eigenvector matrix Q can be formed. The eigenvectors are inserted in column format.

$$Q = \begin{bmatrix} \bar{q}_1 & \bar{q}_2 \end{bmatrix} \tag{41}$$

$$Q = \begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix} \tag{42}$$

The eigenvectors represent orthogonal mode shapes.

Each eigenvector can be multiplied by an arbitrary scale factor. A mass-normalized eigenvector matrix $\hat{Q}$ can be obtained such that the following orthogonality relations are obtained.

$$\hat{Q}^T M \hat{Q} = I \tag{43}$$

and

$$\hat{Q}^T K \hat{Q} = \Omega \tag{44}$$

where

| | |
|---|---|
| $I$ | is the identity matrix |
| $\Omega$ | is a diagonal matrix of eigenvalues |

The superscript T represents transpose.

Note the mass-normalized forms

$$\hat{Q} = \begin{bmatrix} \hat{v}_1 & \hat{w}_1 \\ \hat{v}_2 & \hat{w}_2 \end{bmatrix} \tag{45}$$

$$\hat{Q}^T = \begin{bmatrix} \hat{v}_1 & \hat{v}_2 \\ \hat{w}_1 & \hat{w}_2 \end{bmatrix} \tag{46}$$

Rigorous proof of the orthogonality relationships is beyond the scope of this tutorial.

Further discussion is given in References 1 and 2.

Nevertheless, the orthogonality relationships are demonstrated by an example in this tutorial.

Now define a generalize coordinate $\eta(t)$ such that

$$\bar{x} = \hat{Q}\,\bar{\eta} \tag{47}$$

Substitute equation (47) into the equation of motion, equation (11).

$$M\hat{Q}\,\ddot{\bar{\eta}} + K\hat{Q}\,\bar{\eta} = \bar{F} \tag{48}$$

Premultiply by the transpose of the normalized eigenvector matrix.

$$\hat{Q}^T M\hat{Q}\,\ddot{\bar{\eta}} + \hat{Q}^T K\hat{Q}\,\bar{\eta} = \hat{Q}^T \bar{F} \tag{49}$$

The orthogonality relationships yield

$$I\,\ddot{\bar{\eta}} + \Omega\,\bar{\eta} = \hat{Q}^T \bar{F} \tag{50}$$

The equations of motion along with an added damping matrix become

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \ddot{\eta}_1 \\ \ddot{\eta}_2 \end{bmatrix} + \begin{bmatrix} 2\xi_1\omega_1 & 0 \\ 0 & 2\xi_2\omega_2 \end{bmatrix}\begin{bmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \end{bmatrix} + \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix}\begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} \hat{v}_1 & \hat{v}_2 \\ \hat{w}_1 & \hat{w}_2 \end{bmatrix}\begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} \tag{51}$$

Note that the two equations are decoupled in terms of the generalized coordinate.

Equation (51) yields two equations

$$\ddot{\eta}_1 + 2\xi_1\omega_1\dot{\eta}_1 + \omega_1^2\eta_1 = \hat{v}_1 f_1(t) + \hat{v}_2 f_2(t) \tag{52}$$

$$\ddot{\eta}_2 + 2\xi_2\omega_2\dot{\eta}_2 + \omega_2^2\eta_2 = \hat{w}_1 f_1(t) + \hat{w}_2 f_2(t) \tag{53}$$

The equations can be solved in terms of Laplace transforms, or some other differential equation solution method.

Now consider the initial conditions. Recall

$$\bar{x} = \hat{Q}\,\bar{\eta} \tag{54}$$

Thus

$$\bar{x}(0) = \hat{Q}\,\bar{\eta}(0) \tag{55}$$

Premultiply by $\hat{Q}^T M$.

$$\hat{Q}^T M\,\bar{x}(0) = \hat{Q}^T M\hat{Q}\,\bar{\eta}(0) \tag{56}$$

Recall

$$\hat{Q}^T M\,\hat{Q} = I \tag{57}$$

$$\hat{Q}^T M \bar{x}(0) = I \bar{\eta}(0) \tag{58}$$

$$\hat{Q}^T M \bar{x}(0) = \bar{\eta}(0) \tag{59}$$

Finally, the transformed initial displacement is

$$\bar{\eta}(0) = \hat{Q}^T M \bar{x}(0) \tag{60}$$

Similarly, the transformed initial velocity is

$$\bar{\dot{\eta}}(0) = \hat{Q}^T M \bar{\dot{x}}(0) \tag{61}$$

A basis for a solution is thus derived.

<u>Half-Sine Force</u>

Now consider the special case of a half-sine force applied to mass 2.

$$f_1(t) = 0 \tag{62}$$

$$f_2(t) = B_2 \sin(\beta t) \ , \quad \text{for } t \le T \tag{63}$$

where

$$\beta = \pi / T$$

Thus,

$$\ddot{\eta}_1 + 2\xi_1 \omega_1 \dot{\eta}_1 + \omega_1^2 \eta_1 = \hat{v}_2 B_2 \sin(\beta t) \tag{64}$$

$$\ddot{\eta}_2 + 2\xi_2 \omega_2 \dot{\eta}_2 + \omega_2^2 \eta_2 = \hat{w}_2 B_2 \sin(\beta t) \tag{65}$$

Take the Laplace transform of equation (64).

$$\ddot{\eta}_1 + 2\xi_1 \omega_1 \dot{\eta}_1 + \omega_1^2 \eta_1 = \hat{v}_2 B_2 \sin(\beta t) \tag{66}$$

$$L\left\{\ddot{\eta}_1 + 2\xi_1\omega_1\dot{\eta}_1 + \omega_1^2\eta_1\right\} = L\{\hat{v}_2 B_2 \sin(\beta t)\} \tag{67}$$

$$\begin{aligned} & s^2\hat{\eta}_1(s) - s\eta_1(0) - \dot{\eta}_1(0) \\ & + 2\xi_1\omega_1 s\hat{\eta}_1(s) - 2\xi_1\omega_1\eta_1(0) \\ & + \omega_1^2\hat{\eta}_1(s) = \hat{v}_2 B_2\left\{\frac{\beta}{s^2+\beta^2}\right\} \end{aligned} \tag{68}$$

$$\left\{s^2 + 2\xi_1\omega_1 s + \omega_n^2\right\}\hat{\eta}_1(s) - \left\{s + 2\xi_1\omega_1\right\}\eta_1(0) - \dot{\eta}_1(0) \quad = \quad \hat{v}_2 B_2\left\{\frac{\beta}{s^2+\beta^2}\right\} \tag{69}$$

$$\left\{s^2 + 2\xi_1\omega_1 s + \omega_n^2\right\}\hat{\eta}_1(s) = \hat{v}_2 B_2\left\{\frac{\beta}{s^2+\beta^2}\right\} + \left\{s + 2\xi_1\omega_1\right\}\eta_1(0) + \dot{\eta}_1(0) \tag{70}$$

$$\hat{\eta}_1(s) = \frac{\hat{v}_2 B_2}{\left\{s^2 + 2\xi_1\omega_1 s + \omega_n^2\right\}}\left\{\frac{\beta}{s^2+\beta^2}\right\} + \frac{\left\{s + 2\xi_1\omega_1\right\}\eta_1(0) + \dot{\eta}_1(0)}{\left\{s^2 + 2\xi_1\omega_1 s + \omega_n^2\right\}} \tag{71}$$

The solution is found via References 3 and 4. The inverse Laplace transform for the first modal coordinate is

$$\eta_1(t) =$$

$$+\frac{\hat{v}_2 B_2}{\left[\left(\beta^2-\omega_1^2\right)^2+\left(2\xi_1\beta\omega_1\right)^2\right]}\left\{-\left[2\xi_1\omega_1\beta\right]\cos(\beta t)-\left[\beta^2-\omega_1^2\right]\sin(\beta t)\right\}$$

$$+\frac{\hat{v}_2 B_2\left[\dfrac{\beta}{\omega_{d,1}}\right]\exp\left(-\xi_1\omega_1 t\right)}{\left[\left(\beta^2-\omega_1^2\right)^2+\left(2\xi_1\beta\omega_1\right)^2\right]}\left\{\left[2\xi_1\omega_1\omega_{d,1}\right]\cos\left(\omega_{d,1}t\right)+\left[\beta^2-\omega_1^{\,2}\left(1-2\xi_1^{\,2}\right)\right]\sin\left(\omega_{d,1}t\right)\right\}$$

$$+\exp\left(-\xi_1\omega_1 t\right)\left\{\eta_1(0)\cos\left(\omega_{d,1}t\right)+\left\{\frac{\dot{\eta}_1(0)+\left(\xi_1\omega_1\right)\eta_1(0)}{\omega_{d,1}}\right\}\sin\left(\omega_{d,1}t\right)\right\},$$

for $t \le T$

(72)

Similarly,

$$\eta_2(t) =$$

$$+ \frac{\hat{w}_2 B_2}{\left[\left(\beta^2 - \omega_2^2\right)^2 + (2\xi_2 \beta \omega_2)^2\right]} \left\{ -[2\xi_2\omega_2 \beta]\cos(\beta t) - \frac{1}{\beta}\left[\beta^2 - \omega_2^2\right]\sin(\beta t)\right\}$$

$$+ \frac{\hat{w}_2 B_2 \left[\frac{\beta}{\omega_{d,2}}\right] \exp(-\xi_2\omega_2 t)}{\left[\left(\beta^2 - \omega_2^2\right)^2 + (2\xi_2 \beta \omega_2)^2\right]} \left\{\left[2\xi_2\omega_2\, \omega_{d,2}\right]\cos\left(\omega_{d,2} t\right) + \left[\beta^2 - {\omega_2}^2\left(1 - 2{\xi_2}^2\right)\right]\sin\left(\omega_{d,2} t\right)\right\}$$

$$+ \exp(-\xi_2\omega_2 t)\left\{\eta_2(0)\cos\left(\omega_{d,2} t\right) + \left\{\frac{\dot{\eta}_2(0) + (\xi_2\omega_2)\eta_2(0)}{\omega_{d,2}}\right\}\sin\left(\omega_{d,2} t\right)\right\}$$

for $t \le T$

(73)

The physical displacements are found via

$$\bar{x} = \hat{Q}\,\bar{\eta} \tag{74}$$

<u>Free Vibration</u>

For $t > T$ and $\tau = t - T$

$$\eta_1(t) = \exp(-\xi_1\omega_1\tau)\left\{\eta_1(T)\cos\left(\omega_{d,1}\tau\right) + \left\{\frac{\dot{\eta}_1(T) + (\xi_1\omega_1)\eta_1(T)}{\omega_{d,1}}\right\}\sin\left(\omega_{d,1}\tau\right)\right\} \tag{75}$$

$$\eta_2(t) = \exp(-\xi_2\omega_2\tau)\left\{\eta_2(T)\cos\left(\omega_{d,2}\tau\right) + \left\{\frac{\dot{\eta}_2(T) + (\xi_2\omega_2)\eta_2(T)}{\omega_{d,2}}\right\}\sin\left(\omega_{d,2}\tau\right)\right\} \tag{76}$$

References


1. Bathe, Finite Element Procedures in Engineering Analysis, Prentice-Hall, New Jersey, 1982. Section 12.3.1.

2. Weaver and Johnston, Structural Dynamics by Finite Elements, Prentice-Hall, New Jersey, 1987. Chapter 4.

3. T. Irvine, Table of Laplace Transforms, Vibrationdata, 2000.

4. T. Irvine, Partial Fraction Expansion, Rev F, Vibrationdata, 2010.

## APPENDIX A

Example

Consider the system in Figure 1 with the values in Table A-1.

Assume 5% damping for each mode.  Assume zero initial conditions.

| Table A-1.  Parameters | | |
|---|---|---|
| Variable | Value | Unit |
| $m_1$ | 3.0 | lbf sec^2/in |
| $m_2$ | 2.0 | lbf sec^2/in |
| $k_1$ | 400,000 | lbf/in |
| $k_2$ | 300,000 | lbf/in |
| $k_3$ | 100,000 | lbf/in |
| $B_2$ | 100 | lbf |
| T | 0.011 | sec |

The mass matrix is

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \tag{A-1}$$

The stiffness matrix is

$$K = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} = \begin{bmatrix} 700{,}000 & -300{,}000 \\ -300{,}000 & 400{,}000 \end{bmatrix} \tag{A-2}$$

The analysis is performed using a Matlab script.

```
>> twodof_half_sine_force

 twodof_half_sine_force.m   ver 1.0  January 18, 2012
 by Tom Irvine  Email: tomirvine@aol.com

 This script calculates the response of a two-degree-of-freedom
 system to a half-sine force excitation at dof 2.


 Enter the units system
 1=English  2=metric
 1
 Assume symmetric mass and stiffness matrices.
 Select input mass unit
  1=lbm  2=lbf sec^2/in
 2
 stiffness unit = lbf/in

 Select file input method
   1=file preloaded into Matlab
   2=Excel file
1

 Mass Matrix
 Enter the matrix name:  m_two

 Stiffness Matrix
 Enter the matrix name:  k_two
 Input Matrices

mass =

     3     0
     0     2


stiff =

      700000     -300000
     -300000      400000

 Natural Frequencies
 No.      f(Hz)
1.        48.552
2.        92.839

  Modes Shapes (column format)

ModeShapes =

    0.3797   -0.4349
    0.5326    0.4651
```

```
Enter the damping ratio for mode 1 0.05
Enter the damping ratio for mode 2 0.05

Particpation Factors =

    2.204
  -0.3746

Enter the force amplitude (lbf)  100
Enter the half-sine pulse duration (sec)   0.011

Enter the  first initial displacement (in)  0
Enter the second initial displacement (in)  0

Enter the  first initial velocity (in/sec)  0
Enter the second initial velocity (in/sec)  0

Enter the sample rate (samples/sec) 10000

Enter the analysis duration (sec) 0.15

dof 1 displacement (in)
max= 0.0003287
min= -0.0003149

dof 2 displacement (in)
max= 0.0005005
min= -0.0003728
```

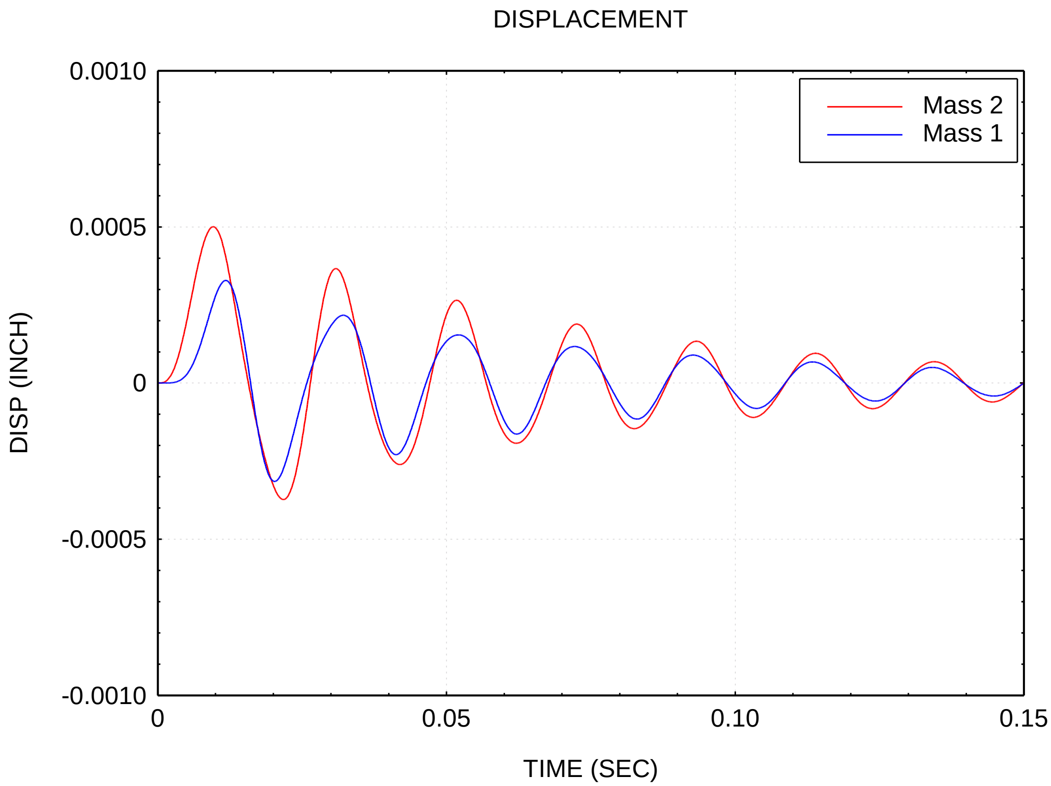
DISPLACEMENT
0.0010
0.0005
0
-0.0005
-0.0010
DISP (INCH)
0
0.05
0.10
0.15
TIME (SEC)
Mass 2
Mass 1

Figure A-1.

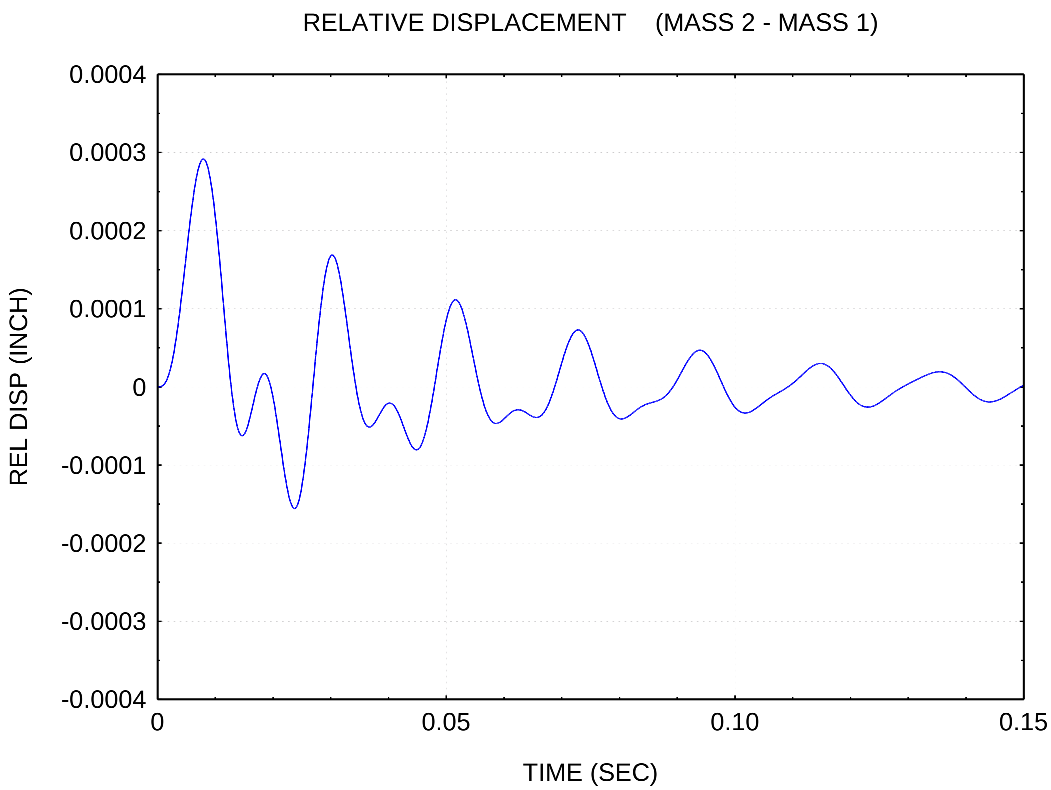
RELATIVE DISPLACEMENT (MASS 2 - MASS 1)
0.0004
0.0003
0.0002
0.0001
0
-0.0001
-0.0002
-0.0003
-0.0004
REL DISP (INCH)
0
0.05
0.10
0.15
TIME (SEC)

Figure A-2.

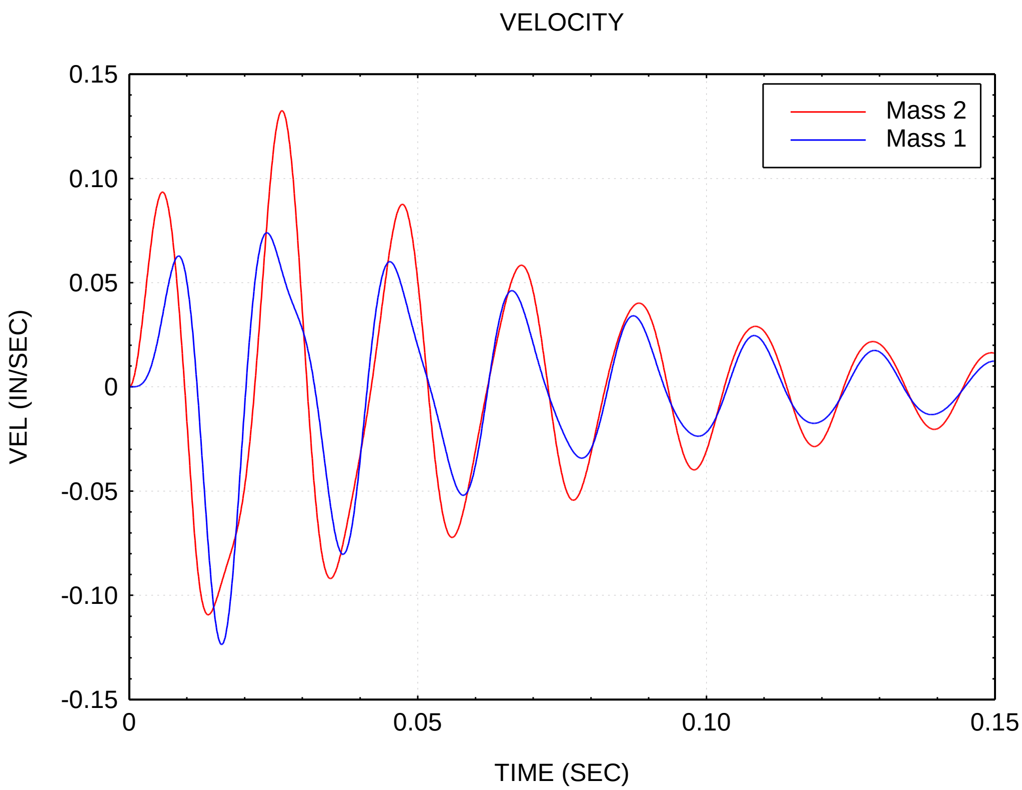
VELOCITY
Mass 2
Mass 1
VEL (IN/SEC)
0.15
0.10
0.05
0
-0.05
-0.10
-0.15
0
0.05
0.10
0.15
TIME (SEC)

Figure A-3.

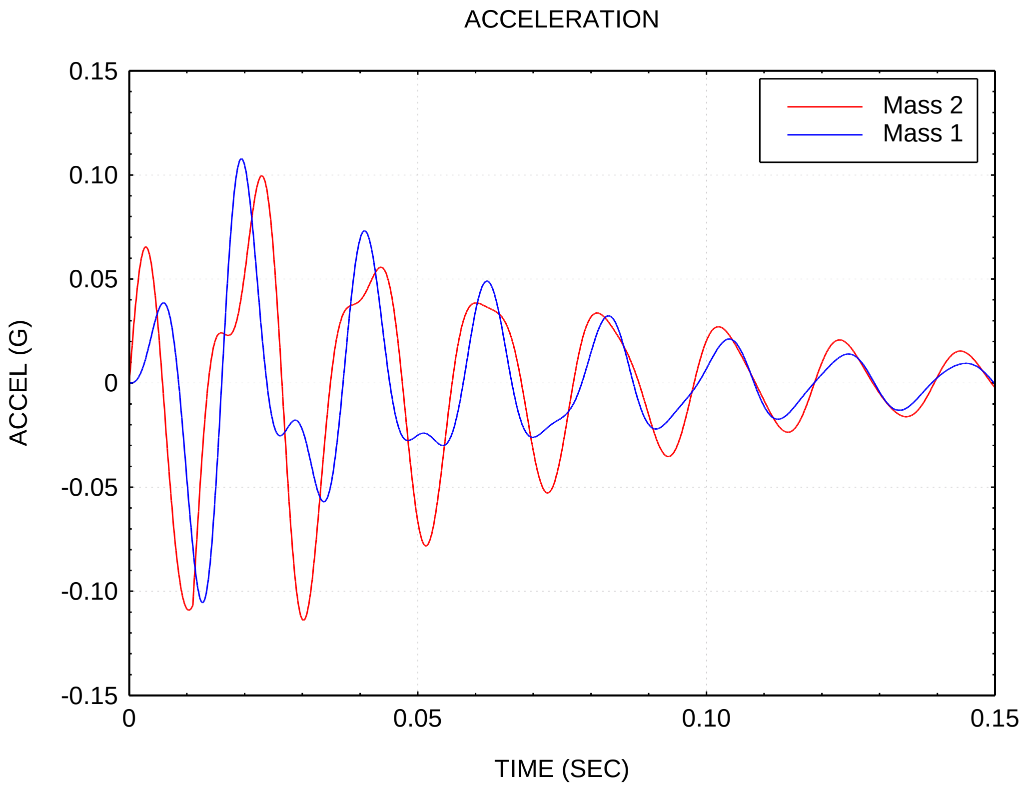
ACCELERATION
0.15
0.10
0.05
0
-0.05
-0.10
-0.15
ACCEL (G)
0
0.05
0.10
0.15
TIME (SEC)
Mass 2
Mass 1

Figure A-4.