THE RESPONSE OF A TWO-DEGREE-OF-FREEDOM SYSTEM
SUBJECTED TO A HALF-SINE BASE INPUT PULSE

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Two-degree-of-freedom System, Modal Analysis

The method of generalized coordinates is demonstrated by an example. Consider the system in Figure 1.

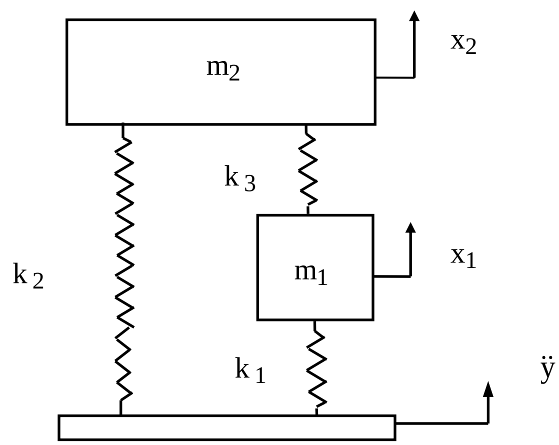

Figure 1.

The system also has damping, but it is modeled as modal damping.

A free-body diagram of mass 1 is given in Figure 2. A free-body diagram of mass 2 is given in Figure 3.

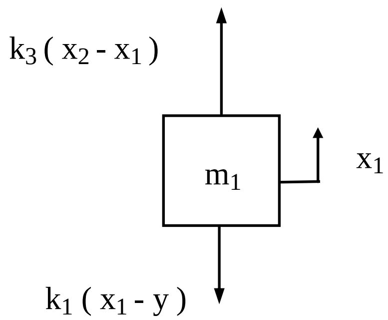

Figure 2.

Determine the equation of motion for mass 1.

$$\Sigma F = m_1 \ddot{x}_1 \tag{1}$$

$$m_1 \ddot{x}_1 = k_3 (x_2 - x_1) - k_1 (x_1 - y) \tag{2}$$

$$m_1 \ddot{x}_1 + k_1 x_1 - k_3 (x_2 - x_1) = k_1 y \tag{3}$$

$$m_1 \ddot{x}_1 + k_1 x_1 + k_3 (x_1 - x_2) = k_1 y \tag{4}$$

$$m_1 \ddot{x}_1 + (k_1 + k_3) x_1 - k_3 x_2 = k_1 y \tag{5}$$

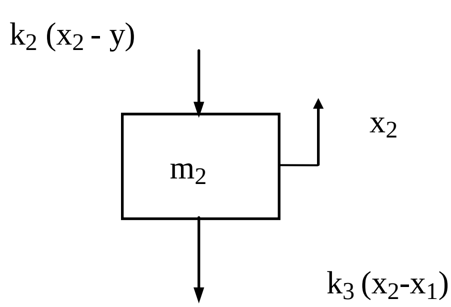

Figure 3.

Derive the equation of motion for mass 2.

$$\Sigma F = m_2 \ddot{x}_2 \tag{6}$$

$$m_2 \ddot{x}_2 = -k_3 (x_2 - x_1) - k_2 (x_2 - y) \tag{7}$$

$$m_2 \ddot{x}_2 + k_2 x_2 + k_3 (x_2 - x_1) = k_2 y \tag{8}$$

$$m_2 \ddot{x}_2 + (k_2 + k_3) x_2 - k_3 x_1 = k_2 y \tag{9}$$

Assemble the equations in matrix form.

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_3 & -k_3 \\ -k_3 & k_2 + k_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} k_1 y \\ k_2 y \end{bmatrix} \tag{10}$$

Define a relative displacement z such that

$$x_1 = z_1 + y \tag{11}$$

$$x_2 = z_2 + y \tag{12}$$

Substitute equations (11) and (12) into (10).

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{z}_1 + \ddot{y} \\ \ddot{z}_2 + \ddot{y} \end{bmatrix} + \begin{bmatrix} k_1 + k_3 & -k_3 \\ -k_3 & k_2 + k_3 \end{bmatrix} \begin{bmatrix} z_1 + y \\ z_2 + y \end{bmatrix} = \begin{bmatrix} k_1 y \\ k_2 y \end{bmatrix} \tag{13}$$

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{z}_1 \\ \ddot{z}_2 \end{bmatrix} + \begin{bmatrix} m_1 \ddot{y} \\ m_2 \ddot{y} \end{bmatrix} + \begin{bmatrix} k_1 + k_3 & -k_3 \\ -k_3 & k_2 + k_3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_3 & -k_3 \\ -k_3 & k_2 + k_3 \end{bmatrix} \begin{bmatrix} y \\ y \end{bmatrix} = \begin{bmatrix} k_1 y \\ k_2 y \end{bmatrix} \tag{14}$$

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{z}_1 \\ \ddot{z}_2 \end{bmatrix} + \begin{bmatrix} m_1 \ddot{y} \\ m_2 \ddot{y} \end{bmatrix} + \begin{bmatrix} k_1 + k_3 & -k_3 \\ -k_3 & k_2 + k_3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} k_1 y \\ k_2 y \end{bmatrix} = \begin{bmatrix} k_1 y \\ k_2 y \end{bmatrix} \tag{15}$$

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{z}_1 \\ \ddot{z}_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_3 & -k_3 \\ -k_3 & k_2 + k_3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} -m_1 \ddot{y} \\ -m_2 \ddot{y} \end{bmatrix} \tag{16}$$

<u>Decoupling</u>

Equation (16) is coupled via the stiffness matrix. An intermediate goal is to decouple the equation.

Simplify,

$$M\,\bar{\ddot{z}} + K\,\bar{z} = \bar{F} \tag{17}$$

where

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \tag{18}$$

$$K = \begin{bmatrix} k_1 + k_3 & -k_3 \\ -k_3 & k_2 + k_3 \end{bmatrix} \tag{19}$$

$$\bar{z} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \tag{20}$$

$$\bar{F} = \begin{bmatrix} -m_1 \ddot{y} \\ -m_2 \ddot{y} \end{bmatrix} \tag{21}$$

Consider the homogeneous form of equation (17).

$$M\,\bar{\ddot{z}} + K\,\bar{z} = \bar{0} \tag{22}$$

Seek a solution of the form

$$\bar{z} = \bar{q}\exp(\,j\omega t) \tag{23}$$

The q vector is the generalized coordinate vector.

Note that

$$\dot{\bar{z}} = j\omega\,\bar{q}\exp(j\omega t) \tag{24}$$

$$\ddot{\bar{z}} = -\omega^2\,\bar{q}\exp(j\omega t) \tag{25}$$

Substitute equations (23) through (25) into equation (22).

$$-\omega^2 M\,\bar{q}\exp(j\omega t) + K\bar{q}\exp(j\omega t) = \bar{0} \tag{26}$$

$$\left\{-\omega^2 M\,\bar{q} + K\bar{q}\right\}\exp(j\omega t) = \bar{0} \tag{27}$$

$$-\omega_n{}^2 M\,\bar{q} + K\bar{q} = \bar{0} \tag{28}$$

$$\left\{-\omega^2 M + K\right\}\bar{q} = \bar{0} \tag{29}$$

$$\left\{K - \omega^2 M\right\}\bar{q} = \bar{0} \tag{30}$$

Equation (30) is an example of a generalized eigenvalue problem. The eigenvalues can be found by setting the determinant equal to zero.

$$\det\left\{K - \omega^2 M\right\} = 0 \tag{31}$$

$$\det\left\{\begin{bmatrix} k_1 + k_3 & -k_3 \\ -k_3 & k_2 + k_3 \end{bmatrix} - \omega^2\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\right\} = 0 \tag{32}$$

$$\det\begin{bmatrix} (k_1 + k_3) - \omega^2 m_1 & -k_3 \\ -k_3 & (k_2 + k_3) - \omega^2 m_2 \end{bmatrix} = 0 \tag{33}$$

$$\left[(k_1 + k_3) - \omega^2 m_1\right]\left[(k_2 + k_3) - \omega^2 m_2\right] - k_3{}^2 = 0 \tag{34}$$

$$\omega^4 m_1 m_2 - \omega^2\left[m_1(k_2 + k_3) + m_2(k_1 + k_3)\right] - k_3{}^2 = 0 \tag{35}$$

The eigenvalues are the roots of the polynomial.

$$\omega_1{}^2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \tag{36}$$

$$\omega_2{}^2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \tag{37}$$

where

$$a = m_1 m_2 \tag{38}$$

$$b = -\left[m_1\left(k_2 + k_3\right) + m_2\left(k_1 + k_3\right)\right] \tag{39}$$

$$c = -k_3{}^2 \tag{40}$$

The eigenvectors are found via the following equations.

$$\left\{K - \omega_1{}^2 M\right\}\bar{q}_1 = \bar{0} \tag{41}$$

$$\left\{K - \omega_2{}^2 M\right\}\bar{q}_2 = \bar{0} \tag{42}$$

where

$$\bar{q}_1 = \begin{bmatrix} q_{11} \\ q_{12} \end{bmatrix} \tag{34}$$

$$\bar{q}_2 = \begin{bmatrix} q_{21} \\ q_{22} \end{bmatrix} \tag{44}$$

An eigenvector matrix Q can be formed. The eigenvectors are inserted in column format.

$$Q = \left[\ \bar{q}_1 \quad | \quad \bar{q}_2\ \right] \tag{45}$$

$$Q = \begin{bmatrix} q_{11} & q_{21} \\ q_{12} & q_{22} \end{bmatrix} \tag{46}$$

The eigenvectors represent orthogonal mode shapes.

Each eigenvector can be multiplied by an arbitrary scale factor. A mass-normalized eigenvector matrix $\hat{Q}$ can be obtained such that the following orthogonality relations are obtained.

$$\hat{Q}^T M \hat{Q} = I \tag{47}$$

and

$$\hat{Q}^T K \hat{Q} = \Omega \tag{48}$$

where

superscript T represents transpose

I is the identity matrix

Ω is a diagonal matrix of eigenvalues

Note that

$$\hat{Q} = \begin{bmatrix} \hat{q}_{11} & \hat{q}_{12} \\ \hat{q}_{21} & \hat{q}_{22} \end{bmatrix} \tag{49a}$$

$$\hat{Q}^T = \begin{bmatrix} \hat{q}_{11} & \hat{q}_{21} \\ \hat{q}_{12} & \hat{q}_{22} \end{bmatrix} \tag{49b}$$

Rigorous proof of the orthogonality relationships is beyond the scope of this tutorial. Further discussion is given in the references.

Nevertheless, the orthogonality relationships are demonstrated by an example in this tutorial.

Now define a modal coordinate $\eta(t)$ such that

$$\bar{z} = \hat{Q}\ \bar{\eta} \tag{50a}$$

$$z_1 = \hat{q}_{11}\,\eta_1 + \hat{q}_{12}\,\eta_2 \tag{50b}$$

$$z_2 = \hat{q}_{21}\,\eta_1 + \hat{q}_{22}\,\eta_2 \tag{50c}$$

Recall

$$x_1 = z_1 + y \tag{51a}$$

$$x_2 = z_2 + y \tag{51b}$$

The displacement terms are

$$x_1 = y + \hat{q}_{11}\,\eta_1 + \hat{q}_{12}\,\eta_2 \tag{51a}$$

$$x_2 = y + \hat{q}_{21}\,\eta_1 + \hat{q}_{22}\,\eta_2 \tag{52b}$$

The velocity terms are

$$\dot{x}_1 = \dot{y} + \hat{q}_{11}\,\dot{\eta}_1 + \hat{q}_{12}\,\dot{\eta}_2 \tag{53a}$$

$$\dot{x}_2 = \dot{y} + \hat{q}_{21}\,\dot{\eta}_1 + \hat{q}_{22}\,\dot{\eta}_2 \tag{53b}$$

The acceleration terms are

$$\ddot{x}_1 = \ddot{y} + \hat{q}_{11}\,\ddot{\eta}_1 + \hat{q}_{12}\,\ddot{\eta}_2 \tag{54a}$$

$$\ddot{x}_2 = \ddot{y} + \hat{q}_{21}\,\ddot{\eta}_1 + \hat{q}_{22}\,\ddot{\eta}_2 \tag{54b}$$

Substitute equation (50a) into the equation of motion, equation (17).

$$M\hat{Q}\,\overline{\ddot{\eta}} + K\hat{Q}\,\overline{\eta} = \overline{F} \tag{55}$$

Premultiply by the transpose of the normalized eigenvector matrix.

$$\hat{Q}^T M\hat{Q}\,\overline{\ddot{\eta}} + \hat{Q}^T K\hat{Q}\,\overline{\eta} = \hat{Q}^T\overline{F} \tag{56}$$

The orthogonality relationships yield

$$I\,\overline{\ddot{\eta}} + \Omega\,\overline{\eta} = \hat{Q}^T\overline{F} \tag{57}$$

For the sample problem, equation (57) becomes

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \ddot{\eta}_1 \\ \ddot{\eta}_2 \end{bmatrix} + \begin{bmatrix} \omega_1^{\,2} & 0 \\ 0 & \omega_2^{\,2} \end{bmatrix}\begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} \hat{q}_{11} & \hat{q}_{21} \\ \hat{q}_{12} & \hat{q}_{22} \end{bmatrix}\begin{bmatrix} -m_1\ddot{y} \\ -m_2\ddot{y} \end{bmatrix} \tag{58}$$

Note that the two equations are decoupled in terms of the modal coordinate.

Now assume modal damping by adding an uncoupled damping matrix.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{\eta}_1 \\ \ddot{\eta}_2 \end{bmatrix} + \begin{bmatrix} 2\xi_1 \omega_1 & 0 \\ 0 & 2\xi_2 \omega_2 \end{bmatrix} \begin{bmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \end{bmatrix} + \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} \hat{q}_{11} & \hat{q}_{21} \\ \hat{q}_{12} & \hat{q}_{22} \end{bmatrix} \begin{bmatrix} -m_1 \ddot{y} \\ -m_2 \ddot{y} \end{bmatrix}$$

(59)

Now consider the initial conditions. Recall

$$\bar{z} = \hat{Q} \ \bar{\eta} \tag{60}$$

Thus,

$$\bar{z}(0) = \hat{Q} \ \bar{\eta}(0) \tag{61}$$

Premultiply by $\hat{Q}^T \ M$.

$$\hat{Q}^T \ M \bar{z}(0) = \hat{Q}^T \ M \hat{Q} \eta(0) \tag{62}$$

Recall

$$\hat{Q}^T M \hat{Q} = I \tag{63}$$

$$\hat{Q}^T \ M \bar{z}(0) = I \ \eta(0) \tag{64}$$

$$\hat{Q}^T \ M \bar{z}(0) = \ \eta(0) \tag{65}$$

Finally, the transformed initial displacement is

$$\eta(0) = \hat{Q}^T \ M \bar{z}(0) \tag{66}$$

Similarly, the transformed initial velocity is

$$\dot{\eta}(0) = \hat{Q}^T \ M \dot{\bar{z}}(0) \tag{67}$$

The product of the first two matrices on the left side of equation (59) equals a vector of participation factors.

$$\begin{bmatrix} -\Gamma_1 \\ -\Gamma_2 \end{bmatrix} = \begin{bmatrix} \hat{q}_{11} & \hat{q}_{21} \\ \hat{q}_{12} & \hat{q}_{22} \end{bmatrix} \begin{bmatrix} m_{11} \\ m_{22} \end{bmatrix} \tag{68}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{\eta}_1 \\ \ddot{\eta}_2 \end{bmatrix} + \begin{bmatrix} 2\xi_1 \omega_1 & 0 \\ 0 & 2\xi_2 \omega_2 \end{bmatrix} \begin{bmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \end{bmatrix} + \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} -\Gamma_1 \ddot{y} \\ -\Gamma_2 \ddot{y} \end{bmatrix} \tag{69}$$

Equation (69) can be solved in terms of Laplace transforms.

Half-Sine Base Input

The base excitation function is:

$$\ddot{y}(t) = \begin{cases} A \sin\left[\dfrac{\pi \ t}{T}\right] \ , & 0 \le t \le T \\ 0 \ , & t > T \end{cases} \tag{70}$$

where

A = acceleration amplitude

T = duration

The equation of motion becomes

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{\eta}_1 \\ \ddot{\eta}_2 \end{bmatrix} + \begin{bmatrix} 2\xi_1 \omega_1 & 0 \\ 0 & 2\xi_2 \omega_2 \end{bmatrix} \begin{bmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \end{bmatrix} + \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} -\Gamma_1 \ddot{y} \\ -\Gamma_2 \ddot{y} \end{bmatrix} \tag{71}$$

$$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}\ddot{\eta}_1\\\ddot{\eta}_2\end{bmatrix} + \begin{bmatrix}2\xi_1\omega_1 & 0\\0 & 2\xi_2\omega_2\end{bmatrix}\begin{bmatrix}\dot{\eta}_1\\\dot{\eta}_2\end{bmatrix} + \begin{bmatrix}\omega_1{}^2 & 0\\0 & \omega_2{}^2\end{bmatrix}\begin{bmatrix}\eta_1\\\eta_2\end{bmatrix} = \begin{bmatrix}-\Gamma_1 A\sin(\pi\ t/T)\\-\Gamma_2 A\sin(\pi\ t/T)\end{bmatrix} \tag{72}$$

The equation of motion for mode i is:

$$\ddot{\eta}_i + 2\xi\omega_n\dot{\eta}_i + \omega_n{}^2\eta_i = -\Gamma_i\ A\sin(\pi\ t/T), \qquad 0 \le t \le T \tag{73}$$

Let

$$\omega = \pi / T \tag{74}$$

The solution to equation (73) is given in Reference 2.

$$\eta_i = \frac{A\Gamma_i}{\left[\left(\omega^2-\omega_i{}^2\right)^2 + \left(2\xi_i\,\omega\omega_i\right)^2\right]}\left[\left(2\xi_i\omega\omega_i\right)\cos(\omega t) + \left(\omega^2-\omega_i{}^2\right)\sin(\omega t)\right]$$

$$-\frac{A\Gamma_i\,\dfrac{\omega}{\omega_{d,i}}\left[\exp\left(-\xi_i\omega_i t\right)\right]}{\left[\left(\omega^2-\omega_i{}^2\right)^2 + \left(2\xi_i\,\omega\omega_i\right)^2\right]}\left[\left(2\xi_i\omega_i\omega_{d,i}\right)\cos\left(\omega_{d,i}t\right) + \left(\omega^2-\omega_i{}^2\left(1-2\xi_i{}^2\right)\right)\sin\left(\omega_{d,i}t\right)\right],$$

$$0 \le t \le T \tag{75}$$

$$\dot{\eta}_i = \frac{\omega A \Gamma_i}{\left[\left(\omega^2 - \omega_i{}^2\right)^2 + \left(2\xi_i \, \omega\omega_i\right)^2\right]}\left[-\left(2\xi_i\omega\omega_i\right)\sin(\omega t) + \left(\omega^2 - \omega_i{}^2\right)\cos(\omega t)\right]$$

$$- \frac{A\Gamma_i \, \omega\left[\exp\left(-\xi_i\omega_i t\right)\right]}{\left[\left(\omega^2 - \omega_i{}^2\right)^2 + \left(2\xi_i \, \omega\omega_i\right)^2\right]}\left[-\left(2\xi_i\omega_i\omega_{d,i}\right)\sin\left(\omega_{d,i} t\right) + \left(\omega^2 - \omega_i{}^2\left(1 - 2\xi_i{}^2\right)\right)\cos\left(\omega_{d,i} t\right)\right]$$

$$+ \frac{A\Gamma_i \, \xi_i \, \frac{\omega_i \omega}{\omega_{d,i}}\left[\exp\left(-\xi_i\omega_i t\right)\right]}{\left[\left(\omega^2 - \omega_i{}^2\right)^2 + \left(2\xi_i \, \omega\omega_i\right)^2\right]}\left[\left(2\xi_i\omega_i\omega_{d,i}\right)\cos\left(\omega_{d,i} t\right) + \left(\omega^2 - \omega_i{}^2\left(1 - 2\xi_i{}^2\right)\right)\sin\left(\omega_{d,i} t\right)\right],$$

,

$$0 \le t \le T$$

(76)

The relative acceleration can then be found from

$$\ddot{\eta}_i = -2\xi\omega_n\dot{\eta}_i - \omega_n{}^2\eta_i - \Gamma_i \; A\sin(\pi \; t/T) \;\; , \quad 0 \le t \le T \tag{77}$$

$$\ddot{\eta}_i = -2\xi\omega_n\dot{\eta}_i - \omega_n{}^2\eta_i \;\; , \quad t > 0 \tag{78}$$

The physical relative displacement is then calculated per

$$\bar{z} = \hat{Q} \; \bar{\eta} \tag{79}$$

The physical relative velocity and relative acceleration terms can then be calculated using the appropriate derivatives.

The physical absolute acceleration is then

$$\ddot{x}_i = \ddot{z}_i + \ddot{y} \tag{80}$$

The free vibration response for t > T can then be calculated via a similar approach using Reference 4.

References

1. T. Irvine, The Generalized Coordinate Method for Discrete Systems, Revision D, Vibrationdata, 2010.

2. T. Irvine, Response of a Single-degree-of-freedom System Subjected to a Classical Pulse Base Excitation, Vibrationdata, 1999.

3. T. Irvine, Effective Modal Mass & Modal Participation Factors, Revision E, Vibrationdata, 2010.

4. T. Irvine, Free Vibration of a Single-Degree-of-Freedom System, Revision B, Vibrationdata, 2005.

APPENDIX A

EXAMPLE 1

Normal Modes Analysis

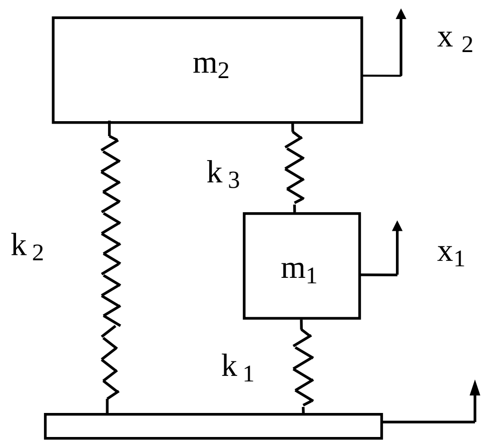

Figure A-1.

Consider the system in Figure A-1. Assign the values in Table A-1.

Table A-1. Parameters		
Variable	Value	Unit
m_1	1158	lbm
	3.0	lbf sec^2/in
m_2	772	lbm
	2.0	lbf sec^2/in
k_1	400,000	lbf/in
k_2	300,000	lbf/in
k_3	100,000	lbf/in

Furthermore, assume

1. Each mode has a damping value of 5%.
2. Zero initial conditions

Next, assume that the base input function is a 10 G, 10 msec half-sine pulse.

Solve for the acceleration response time histories.   The homogeneous, undamped problem is

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{z}_1 \\ \ddot{z}_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_3 & -k_3 \\ -k_3 & k_2 + k_3 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} -m_1\ddot{y} \\ -m_2\ddot{y} \end{bmatrix} \tag{A-2}$$

$$\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} \ddot{z}_1 \\ \ddot{z}_2 \end{bmatrix} + \begin{bmatrix} 500{,}000 & -100{,}000 \\ -100{,}000 & 400{,}000 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{A-3}$$

The eigenvalue problem is

$$\begin{bmatrix} 500{,}000 - 2\omega^2 & -100{,}000 \\ -100{,}000 & 400{,}000 - \omega^2 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{A-4}$$

The analysis is performed using Matlab script: two_dof_frf.m

```
>> two_dof_frf

   two_dof_frf.m   ver 1.7  October 18, 2011
   by Tom Irvine   Email: tomirvine@aol.com

 This program finds the eigenvalues and eigenvectors for a
 two-degree-of-freedom system.
 The equation of motion is:   M (d^2x/dt^2) + K x = 0

 It also finds the frequency response function for base excitation
 and the response to applied excitation.

 Select units: 1=English  2=metric
 1
 Assume symmetric mass and stiffness matrices.

 Enter m11 (lbm)
 1158
 Enter m22 (lbm)
 772
```

```
 Enter k11 (lbf/in)
 500000
 Enter k12 (lbf/in)
 -100000
 Enter k22 (lbf/in)
 400000

 Enter modal damping ratio 1
 0.05
 Enter modal damping ratio 2
 0.05

 The mass matrix is

m =

     3     0
     0     2


 The stiffness matrix is

k =

      500000     -100000
     -100000      400000


  Natural Frequencies =

     59.39 Hz
      75.9 Hz

  Modes Shapes (column format) =

      -0.4792          -0.322
      -0.3943          0.5869

 Participation Factors =

    -2.226
    0.2079

 Effective Modal Mass =

      1913   lbm
     16.69   lbm

 Total Modal Mass =     1930  lbm
```

```
Apply half-sine base input pulse?
 1=yes  2=no
 1

 Enter amplitude(G) 10

 Enter pulse duration(sec) 0.010

 Enter total analysis duration (sec) 0.1

 Enter sample rate (samples/sec)
 (suggest >      1518)
 5000
```

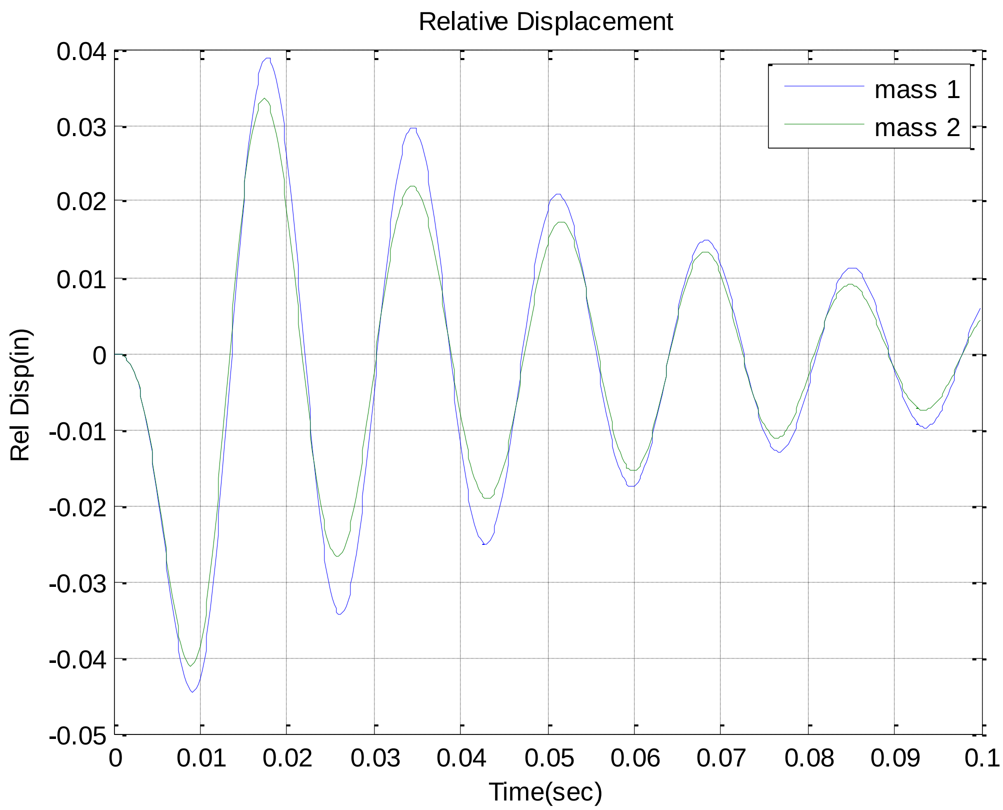
Relative Displacement
0.04
0.03
0.02
0.01
0
-0.01
-0.02
-0.03
-0.04
-0.05
Rel Disp(in)
0
0.01
0.02
0.03
0.04
0.05
0.06
0.07
0.08
0.09
0.1
Time(sec)
mass 1
mass 2

Figure A-2.

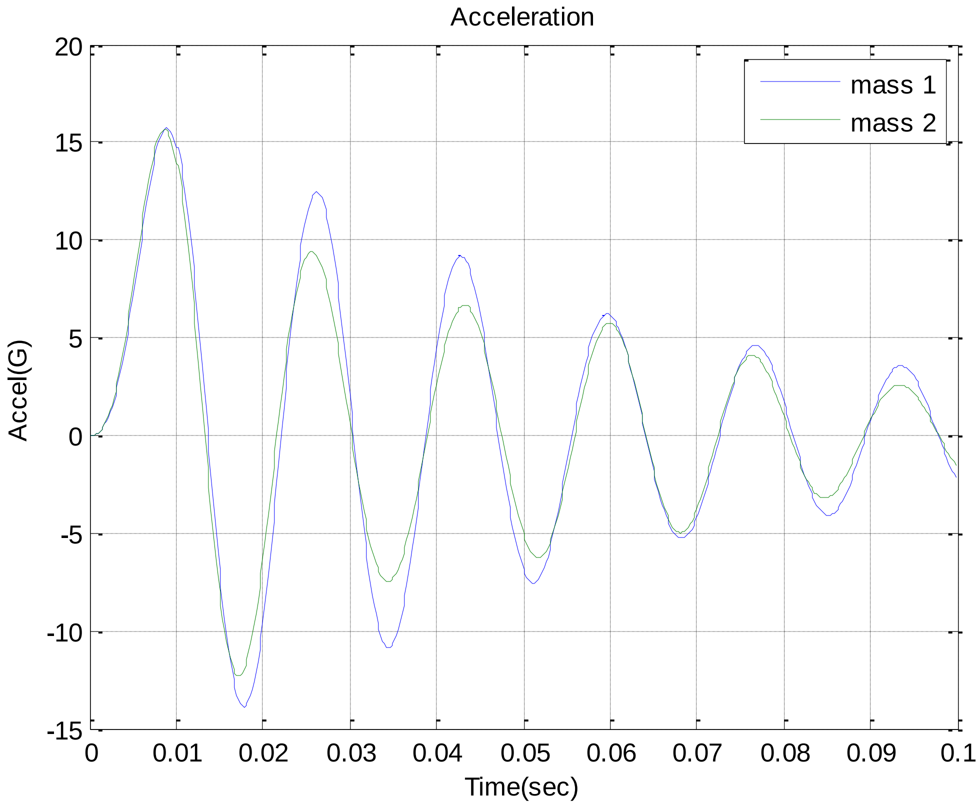
Acceleration
Accel(G)
Time(sec)
mass 1
mass 2
20
15
10
5
0
-5
-10
-15
0
0.01
0.02
0.03
0.04
0.05
0.06
0.07
0.08
0.09
0.1

Figure A-3.

APPENDIX B

Modal Participation Factor

Consider a discrete dynamic system governed by the following equation

$$M\ddot{\bar{x}} + K\bar{x} = \bar{F} \tag{B-1}$$

where

- M is the mass matrix
- K is the stiffness matrix
- $\ddot{\bar{x}}$ is the acceleration vector
- $\bar{x}$ is the displacement vector
- $\bar{F}$ is the forcing function or base excitation function

A solution to the homogeneous form of equation (1) can be found in terms of eigenvalues and eigenvectors. The eigenvectors represent vibration modes.

Let ϕ be the eigenvector matrix.

The system's generalized mass matrix $\hat{m}$ is given by

$$\hat{m} = \phi^T M \phi \tag{B-2}$$

Let $\bar{r}$ be the influence vector which represents the displacements of the masses resulting from static application of a unit ground displacement.

Define a coefficient vector $\bar{L}$ as

$$\bar{L} = \phi^T M \bar{r} \tag{B-3}$$

The modal participation factor matrix Γ_i for mode i is

$$\Gamma_i = \frac{\bar{L}_i}{\hat{m}_{ii}} \tag{B-4}$$

The effective modal mass $m_{eff,i}$ for mode i is

$$m_{eff,i} = \frac{\bar{L}_i^{\ 2}}{\hat{m}_{ii}} \tag{B-5}$$