FREE VIBRATION OF A TWO-DEGREE-OF-FREEDOM SYSTEM SUBJECTED TO INITIAL VELOCITY AND DISPLACEMENT Revision C

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Two-degree-of-freedom System

Consider a two-degree-of-freedom system, as shown in Figure 1. Free-body diagrams are shown in Figure 2.



Figure 1.



Figure 2.

Determine the equation of motion for mass 2.

$$\sum F = m_2 \ddot{x}_2 \tag{1}$$

$$m_2 \ddot{x}_2 = c_2 (\dot{x}_1 - \dot{x}_2) + k_2 (x_1 - x_2)$$
⁽²⁾

$$m_2 \ddot{x}_2 + c_2 (\dot{x}_2 - \dot{x}_1) + k_2 (x_2 - x_1) = 0$$
(3)

Determine the equation of motion for mass 1.

$$\sum F = m_1 \ddot{x}_1 \tag{4}$$

$$m_1 \ddot{x}_1 = -c_2(\dot{x}_1 - \dot{x}_2) + c_1(-\dot{x}_1) - k_2(x_1 - x_2) + k_1(-x_1)$$
(5)

$$m_1 \ddot{x}_1 + c_2 (\dot{x}_1 - \dot{x}_2) + c_1 \dot{x}_1 + k_2 (x_1 - x_2) + k_1 x_1 = 0$$
(6)

$$m_1 \ddot{x}_1 + (c_1 + c_2)\dot{x}_1 - c_2\dot{x}_2 + (k_1 + k_2)x_1 - k_2x_2 = 0$$
(7)

Assemble the equations in matrix form.

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
(8)

Represent as

$$\mathbf{M}\,\overline{\ddot{\mathbf{x}}} + \mathbf{C}\,\overline{\dot{\mathbf{x}}} + \mathbf{K}\,\overline{\mathbf{x}} = \mathbf{F} \tag{9}$$

$$\mathbf{M} = \begin{bmatrix} \mathbf{m}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{m}_2 \end{bmatrix} \tag{10}$$

$$\mathbf{C} = \begin{bmatrix} \mathbf{c}_1 + \mathbf{c}_2 & -\mathbf{c}_2 \\ -\mathbf{c}_2 & \mathbf{c}_2 \end{bmatrix} \tag{11}$$

$$\mathbf{K} = \begin{bmatrix} \mathbf{k}_1 + \mathbf{k}_2 & -\mathbf{k}_2 \\ -\mathbf{k}_2 & \mathbf{k}_2 \end{bmatrix}$$
(12)

$$\mathbf{F} = \begin{bmatrix} 0\\0 \end{bmatrix} \tag{13}$$

Consider the undamped, homogeneous form of equation (9).

$$\mathbf{M}\ \overline{\mathbf{\ddot{x}}} + \mathbf{K}\ \overline{\mathbf{x}} = \overline{\mathbf{0}}\tag{14}$$

Seek a solution of the form

$$\overline{\mathbf{x}} = \overline{\mathbf{q}} \exp(\mathbf{j}\omega \mathbf{t}) \tag{15}$$

The q vector is the generalized coordinate vector.

Note that

$$\overline{\dot{\mathbf{x}}} = \mathbf{j}\omega\,\overline{\mathbf{q}}\,\exp(\mathbf{j}\omega\mathbf{t})\tag{16}$$

$$\overline{\ddot{x}} = -\omega^2 \,\overline{q} \exp(j\omega t) \tag{17}$$

Substitute these equations into equation (14).

$$-\omega^2 \mathbf{M} \ \overline{\mathbf{q}} \exp(\mathbf{j}\omega \mathbf{t}) + \mathbf{K} \ \overline{\mathbf{q}} \exp(\mathbf{j}\omega \mathbf{t}) = \overline{\mathbf{0}}$$
(18)

$$\left\{-\omega^2 \mathbf{M} + \mathbf{K}\right\} \overline{\mathbf{q}} \exp(\mathbf{j}\omega \mathbf{t}) = \overline{\mathbf{0}}$$
⁽¹⁹⁾

$$\left\{-\omega^2 \mathbf{M} + \mathbf{K}\right\} \overline{\mathbf{q}} = \overline{\mathbf{0}} \tag{20}$$

$$\left\{ \mathbf{K} - \omega^2 \mathbf{M} \right\} \overline{\mathbf{q}} = \overline{\mathbf{0}} \tag{21}$$

Equation (21) is an example of a generalized eigenvalue problem. The eigenvalues can be found by setting the determinant equal to zero.

$$\det\left\{\mathbf{K} - \boldsymbol{\omega}^{2}\mathbf{M}\right\} = 0 \tag{22}$$

$$\det\left\{ \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} - \omega^2 \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \right\} = 0$$
(23)

$$\det \begin{cases} (k_1 + k_2) - \omega^2 m_1 & -k_2 \\ -k_2 & k_2 - \omega^2 m_2 \end{cases} = 0$$
(24)

$$\left[(k_1 + k_2) - \omega^2 m_1 \right] k_2 - \omega^2 m_2 - k_2^2 = 0$$
⁽²⁵⁾

$$-\omega^4 m_1 m_2 + \omega^2 \left[-m_2 (k_1 + k_2) - m_1 k_2\right] - k_2^2 + k_2 (k_1 + k_2) = 0$$
⁽²⁶⁾

$$-\omega^4 m_1 m_2 + \omega^2 \left[-m_2 (k_1 + k_2) - m_1 k_2 \right] - k_2^2 + k_1 k_2 + k_2^2 = 0$$
(27)

$$-\omega^4 m_1 m_2 + \omega^2 \left[-m_2 (k_1 + k_2) - m_1 k_2 \right] + k_1 k_2 = 0$$
⁽²⁸⁾

The eigenvalues are the roots of the polynomial.

$$\omega_1^2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$
(29)

$$\omega_2^2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$
(30)

where

$$a = m_1 m_2 \tag{31}$$

$$b = [-m_2(k_1 + k_2) - m_1k_2]$$
(32)

$$\mathbf{c} = \mathbf{k}_1 \mathbf{k}_2 \tag{33}$$

The eigenvectors are found via the following equations.

$$\left\{ \mathbf{K} - \omega_1^2 \mathbf{M} \right\} \overline{\mathbf{q}}_1 = \overline{\mathbf{0}} \tag{34}$$

$$\left\{K - \omega_2^2 M\right\} \overline{q}_2 = \overline{0}$$
(35)

where

$$\overline{\mathbf{q}}_{1} = \begin{bmatrix} \mathbf{v}_{1} \\ \mathbf{v}_{2} \end{bmatrix} \tag{36}$$

$$\overline{\mathbf{q}}_2 = \begin{bmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \end{bmatrix} \tag{37}$$

An eigenvector matrix Q can be formed. The eigenvectors are inserted in column format.

$$Q = \begin{bmatrix} \overline{q}_1 | \overline{q}_2 \end{bmatrix}$$
(38)

$$\mathbf{Q} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{w}_1 \\ \mathbf{v}_2 & \mathbf{w}_2 \end{bmatrix} \tag{39}$$

The eigenvectors represent orthogonal mode shapes.

Each eigenvector can be multiplied by an arbitrary scale factor. A mass-normalized eigenvector matrix \hat{Q} can be obtained such that the following orthogonality relations are obtained.

$$\hat{\mathbf{Q}}^{\mathrm{T}} \mathbf{M} \,\hat{\mathbf{Q}} = \mathbf{I} \tag{40}$$

$$\hat{\mathbf{Q}}^{\mathrm{T}} \mathbf{K} \, \hat{\mathbf{Q}} = \boldsymbol{\Omega} \tag{41}$$

where

superscript T represents transpose I is the identity matrix

 Ω is a diagonal matrix of eigenvalues

Note that

$$\mathbf{Q} = \begin{bmatrix} \hat{\mathbf{v}}_1 & \hat{\mathbf{w}}_1 \\ \hat{\mathbf{v}}_2 & \hat{\mathbf{w}}_2 \end{bmatrix}$$
(42)

$$\mathbf{Q}^{\mathrm{T}} = \begin{bmatrix} \hat{\mathbf{v}}_1 & \hat{\mathbf{v}}_2\\ \hat{\mathbf{w}}_1 & \hat{\mathbf{w}}_2 \end{bmatrix}$$
(43)

Rigorous proof of the orthogonality relationships is beyond the scope of this tutorial. Further discussion is given in References 1 and 2.

Now define a modal coordinate $\eta(t)$ such that

$$\overline{\mathbf{x}} = \hat{\mathbf{Q}}\,\overline{\boldsymbol{\eta}} \tag{44}$$

Substitute equation (44) into equation (9).

$$M\hat{Q}\,\overline{\ddot{\eta}} + C\hat{Q}\,\overline{\dot{\eta}} + K\hat{Q}\,\overline{\eta} = F \tag{45}$$

Premultiply by the transpose of the normalized eigenvector matrix.

$$\hat{Q}^{T}M\hat{Q}\,\overline{\ddot{\eta}} + \hat{Q}^{T}C\hat{Q}\,\overline{\dot{\eta}} + \hat{Q}^{T}K\hat{Q}\,\overline{\eta} = \hat{Q}^{T}F$$
(46)

The orthogonality relationships yield

$$I \ \overline{\ddot{\eta}} + \hat{Q}^{T} C \hat{Q} \,\overline{\dot{\eta}} + \Omega \,\overline{\eta} = \hat{Q}^{T} F$$
(47)

Furthermore, the following assumption is made.

$$\hat{Q}^{T}C\hat{Q}\overline{\dot{\eta}} = \begin{bmatrix} 2\xi_{1}\omega_{1} & 0\\ 0 & 2\xi_{2}\omega_{2} \end{bmatrix}$$
(48)

where $\ \xi_i$ is the modal damping ratio for mode i.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{\eta}_{1} \\ \ddot{\eta}_{2} \end{bmatrix} + \begin{bmatrix} 2\xi_{1} \, \omega_{1} & 0 \\ 0 & 2\xi_{2} \, \omega_{2} \end{bmatrix} \begin{bmatrix} \dot{\eta}_{1} \\ \dot{\eta}_{2} \end{bmatrix} + \begin{bmatrix} \omega_{1}^{2} & 0 \\ 0 & \omega_{2}^{2} \end{bmatrix} \begin{bmatrix} \eta_{1} \\ \eta_{2} \end{bmatrix} = \begin{bmatrix} \hat{v}_{1} & \hat{v}_{2} \\ \hat{w}_{1} & \hat{w}_{2} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
(49)
$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{\eta}_{1} \\ \ddot{\eta}_{2} \end{bmatrix} + \begin{bmatrix} 2\xi_{1} \, \omega_{1} & 0 \\ 0 & 2\xi_{2} \, \omega_{2} \end{bmatrix} \begin{bmatrix} \dot{\eta}_{1} \\ \dot{\eta}_{2} \end{bmatrix} + \begin{bmatrix} \omega_{1}^{2} & 0 \\ 0 & \omega_{2}^{2} \end{bmatrix} \begin{bmatrix} \eta_{1} \\ \eta_{2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

(50)

The two equations are now decoupled in terms of the modal coordinate.

$$\ddot{\eta}_1 + 2\xi_1 \omega_1 \dot{\eta}_1 + \omega_1^2 \eta_1 = 0$$
(51)

$$\ddot{\eta}_2 + 2\xi_2 \,\omega_2 \,\dot{\eta}_2 + \omega_2^2 \,\eta_2 = 0 \tag{52}$$

Take the Laplace transform.

$$L\left\{ \ddot{\eta}_{1} + 2\xi_{1}\omega_{1}\dot{\eta}_{1} + \omega_{1}^{2}\eta_{1} \right\} = 0$$
(53)

$$s^{2} \hat{\eta}_{1}(s) - s\eta_{1}(0) - \dot{\eta}_{1}(0) + 2\xi_{1}\omega_{1}s \hat{\eta}_{1}(s) - 2\xi_{1}\omega_{1}\eta_{1}(0) + \omega_{1}^{2} \hat{\eta}_{1}(s) = 0$$
(54)

$$\left\{s^{2} + 2\xi_{1}\omega_{1}s + \omega_{1}^{2}\right\}\hat{\eta}(s) - \dot{\eta}(0) - \left\{s + 2\xi_{1}\omega_{1}\right\}\eta(0) = 0$$
(55)

$$\left\{s^{2} + 2\xi_{1}\omega_{1}s + \omega_{1}^{2}\right\}\hat{\eta}(s) = \dot{\eta}(0) + \left\{s + 2\xi_{1}\omega_{1}\right\}\eta(0)$$
(56)

$$\hat{\eta}(s) = \frac{1}{\left\{s^2 + 2\xi_1\omega_1 s + \omega_1^2\right\}} \dot{\eta}(0) + \frac{\left\{s + 2\xi_1\omega_1\right\}}{\left\{s^2 + 2\xi_1\omega_1 s + \omega_1^2\right\}} \eta(0)$$
(57)

Consider the denominator.

$$s^{2} + 2\xi_{1}\omega_{1}s + \omega_{1}^{2} = (s + \xi_{1}\omega_{1})^{2} + (1 - \xi_{1}^{2})\omega_{1}^{2}$$
(58)

$$s^{2} + 2\xi_{1}\omega_{1}s + \omega_{1}^{2} = (s + \xi_{1}\omega_{1})^{2} + \omega_{d1}^{2}$$
(59)

Let

$$\omega_{d1} = \sqrt{1 - \xi_1^2} \,\omega_1 \tag{60}$$

By substitution,

$$\hat{\eta}(s) = \frac{1}{\left\{ \left(s + \xi_1 \omega_1 \right)^2 + \omega_{d1}^2 \right\}} \dot{\eta}(0) + \frac{\left\{ s + 2\xi_1 \omega_1 \right\}}{\left\{ \left(s + \xi_1 \omega_1 \right)^2 + \omega_{d1}^2 \right\}} \eta(0)$$
(61)

$$\hat{\eta}(s) = \frac{1}{\left\{ \left(s + \xi_1 \omega_1 \right)^2 + \omega_{d1}^2 \right\}} \dot{\eta}(0) + \frac{\left\{ s + 2\xi_1 \omega_1 \right\}}{\left\{ \left(s + \xi_1 \omega_1 \right)^2 + \omega_{d1}^2 \right\}} \eta(0)$$
(62)

Take the Inverse Laplace transform.

$$\eta_{1}(t) = \exp\left(-\xi_{1}\omega_{1}t\right) \left\{ \eta_{1}(0)\cos(\omega_{d1}t) + \frac{1}{\omega_{d1}} \left[\xi_{1}\omega_{1}\eta_{1}(0) + \dot{\eta}_{1}(0)\right]\sin(\omega_{d1}t) \right\}$$
(63)

$$\dot{\eta}_{1}(t) = -\xi_{1} \omega_{1} \exp\left(-\xi_{1} \omega_{1} t\right) \left\{ \eta_{1}(0) \cos\left(\omega_{d1} t\right) + \frac{1}{\omega_{d1}} [\xi_{1}\omega_{1}\eta_{1}(0) + \dot{\eta}_{1}(0)] \sin\left(\omega_{d1} t\right) \right\} + \exp\left(-\xi_{1} \omega_{1} t\right) \left\{ -\omega_{d1}\eta_{1}(0) \sin\left(\omega_{d1} t\right) + [\xi_{1}\omega_{1}\eta_{1}(0) + \dot{\eta}_{1}(0)] \cos\left(\omega_{d1} t\right) \right\}$$

$$(64)$$

$$\dot{\eta}_{1}(t) = \exp\left(-\xi_{1}\omega_{1}t\right)\left\{-\omega_{d1}\eta_{1}(0) + \frac{-\xi_{1}\omega_{1}}{\omega_{d1}}\left[\xi_{1}\omega_{1}\eta_{1}(0) + \dot{\eta}_{1}(0)\right]\right\}\sin(\omega_{d1}t) + \exp\left(-\xi_{1}\omega_{1}t\right)\left\{\dot{\eta}_{1}(0)\cos(\omega_{d1}t)\right\}$$

(65)

$$\dot{\eta}_{1}(t) = \exp\left(-\xi_{1}\omega_{1} t\right) \left\{ \left\{-\omega_{d1} + \frac{-\xi_{1}^{2}\omega_{1}^{2}}{\omega_{d1}}\right\} \eta_{1}(0) + \frac{-\xi_{1}\omega_{1}\dot{\eta}_{1}(0)}{\omega_{d1}} \right\} \sin(\omega_{d1} t) + \exp\left(-\xi_{1}\omega_{1} t\right) \left\{\dot{\eta}_{1}(0)\cos(\omega_{d1} t)\right\}$$
(66)

Similarly,

$$\eta_{2}(t) = \exp(-\xi_{2}\omega_{2}t) \left\{ \eta_{2}(0)\cos(\omega_{d2}t) + \frac{1}{\omega_{d2}} [\xi_{2}\omega_{2}\eta_{2}(0) + \dot{\eta}_{2}(0)]\sin(\omega_{d2}t) \right\}$$
(67)

$$\overline{\mathbf{x}} = \hat{\mathbf{Q}} \,\overline{\boldsymbol{\eta}} \tag{68}$$

The displacements are

$$x_{1}(t) = v_{1}\eta_{1}(t) + w_{1}\eta_{2}(t)$$
(69)

$$x_2(t) = v_2 \eta_{l(t)} + w_2 \eta_2(t)$$
(70)

Now consider the initial conditions. Recall

$$\overline{\mathbf{x}} = \hat{\mathbf{Q}}\,\overline{\boldsymbol{\eta}} \tag{71}$$

Thus

$$\overline{\mathbf{x}}(0) = \hat{\mathbf{Q}} \,\overline{\boldsymbol{\eta}}(0) \tag{72}$$

Premultiply by $\hat{Q}^T M$.

$$\hat{Q}^{T} \mathbf{M} \,\overline{\mathbf{x}}(0) = \hat{Q}^{T} \,\mathbf{M} \,\hat{Q} \,\,\overline{\eta}(0) \tag{73}$$

Recall

 $\hat{Q}^{T} M \hat{Q} = I \tag{74}$

$$\hat{\mathbf{Q}}^{\mathrm{T}} \mathbf{M} \,\overline{\mathbf{x}}(0) = \mathbf{I} \,\overline{\boldsymbol{\eta}}(0) \tag{75}$$

$$\hat{\mathbf{Q}}^{\mathrm{T}} \mathbf{M} \,\overline{\mathbf{x}}(0) = \overline{\boldsymbol{\eta}}(0) \tag{76}$$

Finally, the transformed initial displacement matrix is

$$\overline{\eta}(0) = \hat{Q}^{T} M \overline{x}(0)$$
(77)

Similarly, the transformed initial velocity is

$$\overline{\dot{\eta}}(0) = \hat{Q}^{T} M \,\overline{\dot{x}}(0) \tag{78}$$

A basis for a solution is thus derived.

References

- 1. Bathe, Finite Element Procedures in Engineering Analysis, Prentice-Hall, New Jersey, 1982. Section 12.3.1.
- 2. Weaver and Johnston, Structural Dynamics by Finite Elements, Prentice-Hall, New Jersey, 1987. Chapter 4.
- 3. T. Irvine, Response of a Single-degree-of-freedom System Subjected to a Classical Pulse Base Excitation, Vibrationdata Publications, 1999.

APPENDIX A

Alternate System



Figure A-1.



Figure A-2.

Determine the equation of motion for mass 2.

$$\sum F = m_2 \ddot{x}_2 \tag{A-1}$$

$$m_2 \ddot{x}_2 = c_2 (\dot{x}_1 - \dot{x}_2) + c_3 (-\dot{x}_2) + k_2 (x_1 - x_2) + k_3 (-x_3)$$
(A-2)

$$m_2 \ddot{x}_2 + c_2 (\dot{x}_2 - \dot{x}_1) + c_3 (\dot{x}_2) + k_2 (x_2 - x_1) + k_3 (x_3) = 0$$
(A-3)

$$m_2 \ddot{x}_2 + (c_2 + c_3)\dot{x}_2 - c_2\dot{x}_1 + (k_2 + k_3)x_2 - k_2x_1 = 0$$
(A-4)

Determine the equation of motion for mass 1.

$$\sum F = m_1 \ddot{x}_1 \tag{A-5}$$

$$m_{1}\ddot{x}_{1} = -c_{2}(\dot{x}_{1} - \dot{x}_{2}) + c_{1}(-\dot{x}_{1}) - k_{2}(x_{1} - x_{2}) + k_{1}(-x_{1})$$
(A-6)

$$m_1 \ddot{x}_1 + c_2 (\dot{x}_1 - \dot{x}_2) + c_1 \dot{x}_1 + k_2 (x_1 - x_2) + k_1 x_1 = 0$$
(A-7)

$$m_1 \ddot{x}_1 + (c_1 + c_2)\dot{x}_1 - c_2\dot{x}_2 + (k_1 + k_2)x_1 - k_2x_2 = 0$$
(A-8)

Assemble the equations in matrix form.

$$\begin{bmatrix} m_1 & 0\\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{x}_1\\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} c_1 + c_2 & -c_2\\ -c_2 & c_2 + c_3 \end{bmatrix} \begin{bmatrix} \dot{x}_1\\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2\\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{bmatrix} x_1\\ x_2 \end{bmatrix} = \begin{bmatrix} 0\\ 0 \end{bmatrix}$$
(A-9)

The problem can now be solved as shown in the main text.

APPENDIX B

Example



Assume 5% damping for each mode.

The initial displacement is

$$x_1 = 0.01 \text{ in}$$

 $x_2 = 0.02 \text{ in}$

The initial velocity is zero for each mass.

Assemble the equations in matrix form.

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
(B-1)

$$\begin{bmatrix} 2/386 & 0 \\ 0 & 1/386 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 25,000 & -10,000 \\ -10,000 & 10,000 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
(B-2)

The results are shown in Figure 1.



Figure B-1.

The response was calculated using Matlab script: mdof_free.m

Natural	Frequencies
No.	f(Hz)
1.	199.47
2.	424.51