# FREE VIBRATION OF A TWO-DEGREE-OF-FREEDOM SYSTEM SUBJECTED TO INITIAL VELOCITY AND DISPLACEMENT Revision C 

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## Two-degree-of-freedom System

Consider a two-degree-of-freedom system, as shown in Figure 1. Free-body diagrams are shown in Figure 2.


Figure 1.


Figure 2.

Determine the equation of motion for mass 2 .

$$
\begin{align*}
& \sum \mathrm{F}=\mathrm{m}_{2} \ddot{\mathrm{x}}_{2}  \tag{1}\\
& \mathrm{~m}_{2} \ddot{\mathrm{x}}_{2}=\mathrm{c}_{2}\left(\dot{\mathrm{x}}_{1}-\dot{\mathrm{x}}_{2}\right)+\mathrm{k}_{2}\left(\mathrm{x}_{1}-\mathrm{x}_{2}\right)  \tag{2}\\
& \mathrm{m}_{2} \ddot{\mathrm{x}}_{2}+\mathrm{c}_{2}\left(\dot{\mathrm{x}}_{2}-\dot{\mathrm{x}}_{1}\right)+\mathrm{k}_{2}\left(\mathrm{x}_{2}-\mathrm{x}_{1}\right)=0 \tag{3}
\end{align*}
$$

Determine the equation of motion for mass 1 .

$$
\begin{align*}
& \sum \mathrm{F}=\mathrm{m}_{1} \ddot{\mathrm{x}}_{1}  \tag{4}\\
& \mathrm{~m}_{1} \ddot{\mathrm{x}}_{1}=-\mathrm{c}_{2}\left(\dot{\mathrm{x}}_{1}-\dot{\mathrm{x}}_{2}\right)+\mathrm{c}_{1}\left(-\dot{\mathrm{x}}_{1}\right)-\mathrm{k}_{2}\left(\mathrm{x}_{1}-\mathrm{x}_{2}\right)+\mathrm{k}_{1}\left(-\mathrm{x}_{1}\right) \\
& \mathrm{m}_{1} \ddot{\mathrm{x}}_{1}+\mathrm{c}_{2}\left(\dot{\mathrm{x}}_{1}-\dot{\mathrm{x}}_{2}\right)+\mathrm{c}_{1} \dot{\mathrm{x}}_{1}+\mathrm{k}_{2}\left(\mathrm{x}_{1}-\mathrm{x}_{2}\right)+\mathrm{k}_{1} \mathrm{x}_{1}=0 \\
& \mathrm{~m}_{1} \ddot{\mathrm{x}}_{1}+\left(\mathrm{c}_{1}+\mathrm{c}_{2}\right) \dot{\mathrm{x}}_{1}-\mathrm{c}_{2} \dot{\mathrm{x}}_{2}+\left(\mathrm{k}_{1}+\mathrm{k}_{2}\right) \mathrm{x}_{1}-\mathrm{k}_{2} \mathrm{x}_{2}=0
\end{align*}
$$

Assemble the equations in matrix form.

$$
\left[\begin{array}{cc}
\mathrm{m}_{1} & 0  \tag{8}\\
0 & \mathrm{~m}_{2}
\end{array}\right]\left[\begin{array}{l}
\ddot{\mathrm{x}}_{1} \\
\ddot{\mathrm{x}}_{2}
\end{array}\right]+\left[\begin{array}{cc}
\mathrm{c}_{1}+\mathrm{c}_{2} & -\mathrm{c}_{2} \\
-\mathrm{c}_{2} & \mathrm{c}_{2}
\end{array}\right]\left[\begin{array}{l}
\dot{\mathrm{x}}_{1} \\
\dot{\mathrm{x}}_{2}
\end{array}\right]+\left[\begin{array}{cc}
\mathrm{k}_{1}+\mathrm{k}_{2} & -\mathrm{k}_{2} \\
-\mathrm{k}_{2} & \mathrm{k}_{2}
\end{array}\right]\left[\begin{array}{l}
\mathrm{x}_{1} \\
\mathrm{x}_{2}
\end{array}\right]=\left[\begin{array}{l}
0 \\
0
\end{array}\right]
$$

Represent as

$$
\begin{align*}
& M \overline{\mathrm{x}}+\mathrm{C} \overline{\mathrm{x}}+\mathrm{K} \overline{\mathrm{x}}=\mathrm{F}  \tag{9}\\
& \mathrm{M}=\left[\begin{array}{cc}
\mathrm{m}_{1} & 0 \\
0 & \mathrm{~m}_{2}
\end{array}\right]  \tag{10}\\
& \mathrm{C}=\left[\begin{array}{cc}
c_{1}+c_{2} & -c_{2} \\
-c_{2} & c_{2}
\end{array}\right] \tag{11}
\end{align*}
$$

$$
\begin{gather*}
\mathrm{K}=\left[\begin{array}{cc}
\mathrm{k}_{1}+\mathrm{k}_{2} & -\mathrm{k}_{2} \\
-\mathrm{k}_{2} & \mathrm{k}_{2}
\end{array}\right]  \tag{12}\\
\mathrm{F}=\left[\begin{array}{l}
0 \\
0
\end{array}\right] \tag{13}
\end{gather*}
$$

Consider the undamped, homogeneous form of equation (9).

$$
\begin{equation*}
\mathrm{M} \overline{\mathrm{x}}+\mathrm{K} \overline{\mathrm{x}}=\overline{0} \tag{14}
\end{equation*}
$$

Seek a solution of the form

$$
\begin{equation*}
\overline{\mathrm{x}}=\overline{\mathrm{q}} \exp (\mathrm{j} \omega \mathrm{t}) \tag{15}
\end{equation*}
$$

The q vector is the generalized coordinate vector.
Note that

$$
\begin{align*}
& \overline{\mathrm{x}}=j \omega \overline{\mathrm{q}} \exp (\mathrm{j} \omega \mathrm{t})  \tag{16}\\
& \overline{\mathrm{x}}=-\omega^{2} \overline{\mathrm{q}} \exp (\mathrm{j} \omega \mathrm{t}) \tag{17}
\end{align*}
$$

Substitute these equations into equation (14).

$$
\begin{align*}
& -\omega^{2} M \overline{\mathrm{q}} \exp (j \omega t)+K \overline{\mathrm{q}} \exp (j \omega t)=\overline{0}  \tag{18}\\
& \left\{-\omega^{2} \mathrm{M}+\mathrm{K}\right\} \overline{\mathrm{q}} \exp (j \omega t)=\overline{0}  \tag{19}\\
& \left\{-\omega^{2} \mathrm{M}+\mathrm{K}\right\} \overline{\mathrm{q}}=\overline{0}  \tag{20}\\
& \left\{K-\omega^{2} M\right\} \overline{\mathrm{q}}=\overline{0} \tag{21}
\end{align*}
$$

Equation (21) is an example of a generalized eigenvalue problem. The eigenvalues can be found by setting the determinant equal to zero.

$$
\begin{align*}
& \operatorname{det}\left\{\mathrm{K}-\omega^{2} \mathrm{M}\right\}=0  \tag{22}\\
& \operatorname{det}\left\{\left[\begin{array}{cc}
\mathrm{k}_{1}+\mathrm{k}_{2} & -\mathrm{k}_{2} \\
-\mathrm{k}_{2} & \mathrm{k}_{2}
\end{array}\right]-\omega^{2}\left[\begin{array}{cc}
\mathrm{m}_{1} & 0 \\
0 & \mathrm{~m}_{2}
\end{array}\right]\right\}=0  \tag{23}\\
& \operatorname{det}\left\{\begin{array}{c}
\left(\mathrm{k}_{1}+\mathrm{k}_{2}\right)-\omega^{2} \mathrm{~m}_{1} \\
-\mathrm{k}_{2} \\
\mathrm{k}_{2}-\omega^{2} \mathrm{~m}_{2}
\end{array}\right\}=0  \tag{24}\\
& \left.\left\lfloor\left(\mathrm{k}_{1}+\mathrm{k}_{2}\right)-\omega^{2} \mathrm{~m}_{1}\right] \mathrm{k}_{2}-\omega^{2} \mathrm{~m}_{2}\right]-\mathrm{k}_{2}^{2}=0  \tag{25}\\
& -\omega^{4} \mathrm{~m}_{1} \mathrm{~m}_{2}+\omega^{2}\left[-\mathrm{m}_{2}\left(\mathrm{k}_{1}+\mathrm{k}_{2}\right)-\mathrm{m}_{1} \mathrm{k}_{2}\right]-\mathrm{k}_{2}^{2}+\mathrm{k}_{2}\left(\mathrm{k}_{1}+\mathrm{k}_{2}\right)=0  \tag{26}\\
& -\omega^{4} \mathrm{~m}_{1} \mathrm{~m}_{2}+\omega^{2}\left[-\mathrm{m}_{2}\left(\mathrm{k}_{1}+\mathrm{k}_{2}\right)-\mathrm{m}_{1} \mathrm{k}_{2}\right]-\mathrm{k}_{2}^{2}+\mathrm{k}_{1} \mathrm{k}_{2}+\mathrm{k}_{2}^{2}=0  \tag{27}\\
& -\omega^{4} \mathrm{~m}_{1} \mathrm{~m}_{2}+\omega^{2}\left[-\mathrm{m}_{2}\left(\mathrm{k}_{1}+\mathrm{k}_{2}\right)-\mathrm{m}_{1} \mathrm{k}_{2}\right]+\mathrm{k}_{1} \mathrm{k}_{2}=0 \tag{28}
\end{align*}
$$

The eigenvalues are the roots of the polynomial.

$$
\begin{equation*}
\omega_{1}^{2}=\frac{-\mathrm{b}-\sqrt{\mathrm{b}^{2}-4 \mathrm{ac}}}{2 \mathrm{a}} \tag{29}
\end{equation*}
$$

$$
\begin{equation*}
\omega_{2}^{2}=\frac{-\mathrm{b}+\sqrt{\mathrm{b}^{2}-4 \mathrm{ac}}}{2 \mathrm{a}} \tag{30}
\end{equation*}
$$

where

$$
\begin{align*}
& \mathrm{a}=\mathrm{m}_{1} \mathrm{~m}_{2}  \tag{31}\\
& \mathrm{~b}=\left[-\mathrm{m}_{2}\left(\mathrm{k}_{1}+\mathrm{k}_{2}\right)-\mathrm{m}_{1} \mathrm{k}_{2}\right]  \tag{32}\\
& \mathrm{c}=\mathrm{k}_{1} \mathrm{k}_{2} \tag{33}
\end{align*}
$$

The eigenvectors are found via the following equations.

$$
\begin{align*}
\left\{\mathrm{K}-\omega_{1}^{2} \mathrm{M}\right\} \overline{\mathrm{q}}_{1} & =\overline{0}  \tag{34}\\
\left\{\mathrm{~K}-\omega_{2}^{2} \mathrm{M}\right\} \overline{\mathrm{q}}_{2} & =\overline{0} \tag{35}
\end{align*}
$$

where

$$
\begin{gather*}
\overline{\mathrm{q}}_{1}=\left[\begin{array}{l}
\mathrm{v}_{1} \\
\mathrm{v}_{2}
\end{array}\right]  \tag{36}\\
\overline{\mathrm{q}}_{2}=\left[\begin{array}{l}
\mathrm{w}_{1} \\
\mathrm{w}_{2}
\end{array}\right] \tag{37}
\end{gather*}
$$

An eigenvector matrix Q can be formed. The eigenvectors are inserted in column format.

$$
\begin{align*}
& \mathrm{Q}=\left[\begin{array}{ll}
\overline{\mathrm{q}}_{1} \mid & \overline{\mathrm{q}}_{2}
\end{array}\right]  \tag{38}\\
& \mathrm{Q}=\left[\begin{array}{ll}
\mathrm{v}_{1} & \mathrm{w}_{1} \\
\mathrm{v}_{2} & \mathrm{w}_{2}
\end{array}\right] \tag{39}
\end{align*}
$$

The eigenvectors represent orthogonal mode shapes.
Each eigenvector can be multiplied by an arbitrary scale factor. A mass-normalized eigenvector matrix $\hat{\mathrm{Q}}$ can be obtained such that the following orthogonality relations are obtained.

$$
\begin{align*}
& \hat{\mathrm{Q}}^{\mathrm{T}} \mathrm{M} \hat{\mathrm{Q}}=\mathrm{I}  \tag{40}\\
& \hat{\mathrm{Q}}^{\mathrm{T}} \mathrm{~K} \hat{\mathrm{Q}}=\Omega \tag{41}
\end{align*}
$$

where
superscript T represents transpose
I is the identity matrix
$\Omega$ is a diagonal matrix of eigenvalues

Note that

$$
\begin{align*}
\mathrm{Q} & =\left[\begin{array}{ll}
\hat{\mathrm{v}}_{1} & \hat{\mathrm{w}}_{1} \\
\hat{\mathrm{v}}_{2} & \hat{\mathrm{w}}_{2}
\end{array}\right]  \tag{42}\\
\mathrm{Q}^{\mathrm{T}} & =\left[\begin{array}{cc}
\hat{\mathrm{v}}_{1} & \hat{\mathrm{v}}_{2} \\
\hat{\mathrm{w}}_{1} & \hat{\mathrm{w}}_{2}
\end{array}\right] \tag{43}
\end{align*}
$$

Rigorous proof of the orthogonality relationships is beyond the scope of this tutorial. Further discussion is given in References 1 and 2.

Now define a modal coordinate $\eta(\mathrm{t})$ such that

$$
\begin{equation*}
\bar{x}=\hat{Q} \bar{\eta} \tag{44}
\end{equation*}
$$

Substitute equation (44) into equation (9).

$$
\begin{equation*}
\mathrm{MQ} \overline{\ddot{\eta}}+\mathrm{C} \hat{\mathrm{Q}} \overline{\dot{\eta}}+\mathrm{K} \hat{\mathrm{Q}} \bar{\eta}=\mathrm{F} \tag{45}
\end{equation*}
$$

Premultiply by the transpose of the normalized eigenvector matrix.

$$
\begin{equation*}
\hat{\mathrm{Q}}^{\mathrm{T}} \mathrm{M} \hat{\mathrm{Q}} \overline{\ddot{\eta}}+\hat{\mathrm{Q}}^{\mathrm{T}} \mathrm{C} \hat{\mathrm{Q}} \dot{\bar{\eta}}+\hat{\mathrm{Q}}^{\mathrm{T}} \mathrm{~K} \hat{\mathrm{Q}} \bar{\eta}=\hat{\mathrm{Q}}^{\mathrm{T}} \mathrm{~F} \tag{46}
\end{equation*}
$$

The orthogonality relationships yield

$$
\begin{equation*}
\mathrm{I} \overline{\ddot{\eta}}+\hat{\mathrm{Q}}^{\mathrm{T}} \mathrm{C} \hat{\mathrm{Q}} \overline{\dot{\eta}}+\Omega \bar{\eta}=\hat{\mathrm{Q}}^{\mathrm{T}} \mathrm{~F} \tag{47}
\end{equation*}
$$

Furthermore, the following assumption is made.

$$
\hat{Q}^{\mathrm{T}} \mathrm{C} \hat{\mathrm{Q}} \overline{\dot{\eta}}=\left[\begin{array}{cc}
2 \xi_{1} \omega_{1} & 0  \tag{48}\\
0 & 2 \xi_{2} \omega_{2}
\end{array}\right]
$$

where $\xi_{i}$ is the modal damping ratio for mode i .

$$
\begin{align*}
& {\left[\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right]\left[\begin{array}{l}
\ddot{\eta}_{1} \\
\ddot{\eta}_{2}
\end{array}\right]+\left[\begin{array}{cc}
2 \xi_{1} \omega_{1} & 0 \\
0 & 2 \xi_{2} \omega_{2}
\end{array}\right]\left[\begin{array}{l}
\dot{\eta}_{1} \\
\dot{\eta}_{2}
\end{array}\right]+\left[\begin{array}{cc}
\omega_{1}^{2} & 0 \\
0 & \omega_{2}^{2}
\end{array}\right]\left[\begin{array}{l}
\eta_{1} \\
\eta_{2}
\end{array}\right]=\left[\begin{array}{cc}
\hat{v}_{1} & \hat{v}_{2} \\
\hat{w}_{1} & \hat{w}_{2}
\end{array}\right]\left[\begin{array}{l}
0 \\
0
\end{array}\right]} \\
& {\left[\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right]\left[\begin{array}{c}
\ddot{\eta}_{1} \\
\ddot{\eta}_{2}
\end{array}\right]+\left[\begin{array}{cc}
2 \xi_{1} \omega_{1} & 0 \\
0 & 2 \xi_{2} \omega_{2}
\end{array}\right]\left[\begin{array}{l}
\dot{\eta}_{1} \\
\dot{\eta}_{2}
\end{array}\right]+\left[\begin{array}{cc}
\omega_{1}^{2} & 0 \\
0 & \omega_{2}^{2}
\end{array}\right]\left[\begin{array}{l}
\eta_{1} \\
\eta_{2}
\end{array}\right]=\left[\begin{array}{l}
0 \\
0
\end{array}\right]} \tag{49}
\end{align*}
$$

The two equations are now decoupled in terms of the modal coordinate.

$$
\begin{align*}
& \ddot{\eta}_{1}+2 \xi_{1} \omega_{1} \dot{\eta}_{1}+\omega_{1}^{2} \eta_{1}=0  \tag{51}\\
& \ddot{\eta}_{2}+2 \xi_{2} \omega_{2} \dot{\eta}_{2}+\omega_{2}^{2} \eta_{2}=0 \tag{52}
\end{align*}
$$

Take the Laplace transform.

$$
\begin{equation*}
L\left\{\ddot{\eta}_{1}+2 \xi_{1} \omega_{1} \dot{\eta}_{1}+\omega_{1}^{2} \eta_{1}\right\}=0 \tag{53}
\end{equation*}
$$

$$
\begin{align*}
& s^{2} \hat{\eta}_{1}(s)-s \eta_{1}(0)-\dot{\eta}_{1}(0)+2 \xi_{1} \omega_{1} s \hat{\eta}_{1}(s)-2 \xi_{1} \omega_{1} \eta_{1}(0)+\omega_{1}^{2} \hat{\eta}_{1}(s)=0  \tag{54}\\
& \left\{s^{2}+2 \xi_{1} \omega_{1} s+\omega_{1}^{2}\right\} \hat{\eta}(s)-\dot{\eta}(0)-\left\{s+2 \xi_{1} \omega_{1}\right\} \eta(0)=0  \tag{55}\\
& \left\{s^{2}+2 \xi_{1} \omega_{1} s+\omega_{1}^{2}\right\} \hat{\eta}(s)=\dot{\eta}(0)+\left\{s+2 \xi_{1} \omega_{1}\right\} \eta(0)  \tag{56}\\
& \hat{\eta}(s)=\frac{1}{\left\{s^{2}+2 \xi_{1} \omega_{1} s+\omega_{1}^{2}\right\}} \dot{\eta}(0)+\frac{\left\{s+2 \xi_{1} \omega_{1}\right\}}{\left\{s^{2}+2 \xi_{1} \omega_{1} s+\omega_{1}^{2}\right\}} \eta(0) \tag{57}
\end{align*}
$$

Consider the denominator.

$$
\begin{align*}
& \mathrm{s}^{2}+2 \xi_{1} \omega_{1} \mathrm{~s}+\omega_{1}^{2}=\left(\mathrm{s}+\xi_{1} \omega_{1}\right)^{2}+\left(1-\xi_{1}^{2}\right) \omega_{1}^{2}  \tag{58}\\
& \mathrm{~s}^{2}+2 \xi_{1} \omega_{1} \mathrm{~s}+\omega_{1}^{2}=\left(\mathrm{s}+\xi_{1} \omega_{1}\right)^{2}+\omega_{\mathrm{d} 1}^{2} \tag{59}
\end{align*}
$$

Let

$$
\begin{equation*}
\omega_{\mathrm{d} 1}=\sqrt{1-\xi_{1}^{2}} \omega_{1} \tag{60}
\end{equation*}
$$

By substitution,

$$
\begin{equation*}
\hat{\eta}(\mathrm{s})=\frac{1}{\left.\left\{\left(\mathrm{~s}+\xi_{1} \omega_{1}\right)^{2}+\omega_{\mathrm{d} 1}{ }^{2}\right\}^{\eta}(0)+\frac{\left\{s+2 \xi_{1} \omega_{1}\right\}}{\left\{\left(\mathrm{s}+\xi_{1} \omega_{1}\right)^{2}+\omega_{\mathrm{d} 1}^{2}\right\}^{\eta(0)}}{ }^{\eta}\right) .} \tag{61}
\end{equation*}
$$

$$
\begin{equation*}
\hat{\eta}(s)=\frac{1}{\left\{\left(s+\xi_{1} \omega_{1}\right)^{2}+\omega_{\mathrm{d} 1}{ }^{2}\right\}} \dot{\eta}(0)+\frac{\left\{s+2 \xi_{1} \omega_{1}\right\}}{\left\{\left(s+\xi_{1} \omega_{1}\right)^{2}+\omega_{\mathrm{d} 1}^{2}\right\}^{\eta}} \eta \tag{62}
\end{equation*}
$$

Take the Inverse Laplace transform.

$$
\begin{align*}
& \eta_{1}(t)=\exp \left(-\xi_{1} \omega_{1} t\right)\left\{\eta_{1}(0) \cos \left(\omega_{d 1} t\right)+\frac{1}{\omega_{d 1}}\left[\xi_{1} \omega_{1} \eta_{1}(0)+\dot{\eta}_{1}(0)\right] \sin \left(\omega_{d 1} t\right)\right\}  \tag{63}\\
& \dot{\eta}_{1}(t)=-\xi_{1} \omega_{1} \exp \left(-\xi_{1} \omega_{1} t\right)\left\{\eta_{1}(0) \cos \left(\omega_{d 1} t\right)+\frac{1}{\omega_{d 1}}\left[\xi_{1} \omega_{1} \eta_{1}(0)+\dot{\eta}_{1}(0)\right] \sin \left(\omega_{d 1} t\right)\right\} \\
& +\exp \left(-\xi_{1} \omega_{1} t\right)\left\{-\omega_{d 1} \eta_{1}(0) \sin \left(\omega_{\mathrm{d} 1} \mathrm{t}\right)+\left[\xi_{1} \omega_{1} \eta_{1}(0)+\dot{\eta}_{1}(0)\right] \cos \left(\omega_{\mathrm{d} 1} \mathrm{t}\right)\right\}  \tag{64}\\
& \dot{\eta}_{1}(\mathrm{t})=\exp \left(-\xi_{1} \omega_{1} \mathrm{t}\right)\left\{-\omega_{\mathrm{d} 1} \eta_{1}(0)+\frac{-\xi_{1} \omega_{1}}{\omega_{\mathrm{d} 1}}\left[\xi_{1} \omega_{1} \eta_{1}(0)+\dot{\eta}_{1}(0)\right]\right\} \sin \left(\omega_{\mathrm{d} 1} \mathrm{t}\right) \\
& +\exp \left(-\xi_{1} \omega_{1} t\right)\left\{\dot{\eta}_{1}(0) \cos \left(\omega_{d} \mathrm{t}\right)\right\}  \tag{65}\\
& \dot{\eta}_{1}(\mathrm{t})=\exp \left(-\xi_{1} \omega_{1} \mathrm{t}\right)\left\{\left\{-\omega_{\mathrm{d} 1}+\frac{-\xi_{1}^{2} \omega_{1}^{2}}{\omega_{\mathrm{d} 1}}\right\} \eta_{1}(0)+\frac{-\xi_{1} \omega_{1} \dot{\eta}_{1}(0)}{\omega_{\mathrm{d} 1}}\right\} \sin \left(\omega_{\mathrm{d} 1} \mathrm{t}\right) \\
& +\exp \left(-\xi_{1} \omega_{1} t\right)\left\{\dot{\eta}_{1}(0) \cos \left(\omega_{\mathrm{d}} \mathrm{t}\right)\right\} \tag{66}
\end{align*}
$$

Similarly,

$$
\begin{gather*}
\eta_{2}(t)=\exp \left(-\xi_{2} \omega_{2} t\right)\left\{\eta_{2}(0) \cos \left(\omega_{d 2} t\right)+\frac{1}{\omega_{d 2}}\left[\xi_{2} \omega_{2} \eta_{2}(0)+\dot{\eta}_{2}(0)\right] \sin \left(\omega_{d 2} t\right)\right\} \\
\bar{x}=\hat{Q} \bar{\eta} \tag{68}
\end{gather*}
$$

The displacements are

$$
\begin{align*}
& x_{1}(t)=v_{1} \eta_{1}(t)+w_{1} \eta_{2}(t)  \tag{69}\\
& x_{2}(t)=v_{2} \eta_{1}(t)+w_{2} \eta_{2}(t) \tag{70}
\end{align*}
$$

Now consider the initial conditions. Recall

$$
\begin{equation*}
\bar{x}=\hat{\mathrm{Q}} \bar{\eta} \tag{71}
\end{equation*}
$$

Thus

$$
\begin{equation*}
\overline{\mathrm{x}}(0)=\hat{\mathrm{Q}} \bar{\eta}(0) \tag{72}
\end{equation*}
$$

Premultiply by $\hat{\mathrm{Q}}^{\mathrm{T}} \mathrm{M}$.

$$
\begin{equation*}
\hat{\mathrm{Q}}^{\mathrm{T}} \mathrm{M} \overline{\mathrm{x}}(0)=\hat{\mathrm{Q}}^{\mathrm{T}} \mathrm{M} \hat{\mathrm{Q}} \bar{\eta}(0) \tag{73}
\end{equation*}
$$

Recall

$$
\begin{align*}
& \hat{\mathrm{Q}}^{\mathrm{T}} \mathrm{M} \hat{\mathrm{Q}}=\mathrm{I}  \tag{74}\\
& \hat{\mathrm{Q}}^{\mathrm{T}} \mathrm{M} \overline{\mathrm{x}}(0)=\mathrm{I} \bar{\eta}(0)  \tag{75}\\
& \hat{\mathrm{Q}}^{\mathrm{T}} \mathrm{M} \overline{\mathrm{x}}(0)=\bar{\eta}(0) \tag{76}
\end{align*}
$$

Finally, the transformed initial displacement matrix is

$$
\begin{equation*}
\bar{\eta}(0)=\hat{\mathrm{Q}}^{\mathrm{T}} \mathrm{M} \overline{\mathrm{x}}(0) \tag{77}
\end{equation*}
$$

Similarly, the transformed initial velocity is

$$
\begin{equation*}
\overline{\dot{\eta}}(0)=\hat{\mathrm{Q}}^{\mathrm{T}} \mathrm{M} \overline{\mathrm{x}}(0) \tag{78}
\end{equation*}
$$

A basis for a solution is thus derived.

## References

1. Bathe, Finite Element Procedures in Engineering Analysis, Prentice-Hall, New Jersey, 1982. Section 12.3.1.
2. Weaver and Johnston, Structural Dynamics by Finite Elements, Prentice-Hall, New Jersey, 1987. Chapter 4.
3. T. Irvine, Response of a Single-degree-of-freedom System Subjected to a Classical Pulse Base Excitation, Vibrationdata Publications, 1999.

## APPENDIX A

Alternate System


Figure A-1.
$\mathrm{k}_{3}\left(-\mathrm{x}_{2}\right) \quad \mathrm{c}_{3}\left(-\dot{\mathrm{x}}_{2}\right)$

$\mathrm{k}_{2}\left(\mathrm{x}_{1}-\mathrm{x}_{2}\right) \quad \mathrm{c}_{2}\left(\dot{\mathrm{x}}_{1}-\dot{\mathrm{x}}_{2}\right)$
$-\mathrm{k}_{2}\left(\mathrm{x}_{1}-\mathrm{x}_{2}\right) \quad-\mathrm{c}_{2}\left(\dot{\mathrm{x}}_{1}-\dot{\mathrm{x}}_{2}\right)$


$$
\mathrm{k}_{1}\left(-\mathrm{x}_{1}\right) \quad \mathrm{c}_{1}\left(-\dot{\mathrm{x}}_{1}\right)
$$

Figure A-2.

Determine the equation of motion for mass 2 .

$$
\begin{align*}
& \sum \mathrm{F}=\mathrm{m}_{2} \ddot{\mathrm{x}}_{2}  \tag{A-1}\\
& \mathrm{~m}_{2} \ddot{\mathrm{x}}_{2}=\mathrm{c}_{2}\left(\dot{\mathrm{x}}_{1}-\dot{\mathrm{x}}_{2}\right)+\mathrm{c}_{3}\left(-\dot{\mathrm{x}}_{2}\right)+\mathrm{k}_{2}\left(\mathrm{x}_{1}-\mathrm{x}_{2}\right)+\mathrm{k}_{3}\left(-\mathrm{x}_{3}\right)  \tag{A-2}\\
& \mathrm{m}_{2} \ddot{\mathrm{x}}_{2}+\mathrm{c}_{2}\left(\dot{\mathrm{x}}_{2}-\dot{\mathrm{x}}_{1}\right)+\mathrm{c}_{3}\left(\dot{\mathrm{x}}_{2}\right)+\mathrm{k}_{2}\left(\mathrm{x}_{2}-\mathrm{x}_{1}\right)+\mathrm{k}_{3}\left(\mathrm{x}_{3}\right)=0  \tag{A-3}\\
& \mathrm{~m}_{2} \ddot{\mathrm{x}}_{2}+\left(\mathrm{c}_{2}+\mathrm{c}_{3}\right) \dot{\mathrm{x}}_{2}-\mathrm{c}_{2} \dot{\mathrm{x}}_{1}+\left(\mathrm{k}_{2}+\mathrm{k}_{3}\right) \mathrm{x}_{2}-\mathrm{k}_{2} \mathrm{x}_{1}=0 \tag{A-4}
\end{align*}
$$

Determine the equation of motion for mass 1 .

$$
\begin{align*}
& \sum \mathrm{F}=\mathrm{m}_{1} \ddot{\mathrm{x}}_{1}  \tag{A-5}\\
& \mathrm{~m}_{1} \ddot{\mathrm{x}}_{1}=-\mathrm{c}_{2}\left(\dot{\mathrm{x}}_{1}-\dot{\mathrm{x}}_{2}\right)+\mathrm{c}_{1}\left(-\dot{\mathrm{x}}_{1}\right)-\mathrm{k}_{2}\left(\mathrm{x}_{1}-\mathrm{x}_{2}\right)+\mathrm{k}_{1}\left(-\mathrm{x}_{1}\right)  \tag{A-6}\\
& \mathrm{m}_{1} \ddot{\mathrm{x}}_{1}+\mathrm{c}_{2}\left(\dot{\mathrm{x}}_{1}-\dot{\mathrm{x}}_{2}\right)+\mathrm{c}_{1} \dot{\mathrm{x}}_{1}+\mathrm{k}_{2}\left(\mathrm{x}_{1}-\mathrm{x}_{2}\right)+\mathrm{k}_{1} \mathrm{x}_{1}=0  \tag{A-7}\\
& \mathrm{~m}_{1} \ddot{\mathrm{x}}_{1}+\left(\mathrm{c}_{1}+\mathrm{c}_{2}\right) \dot{\mathrm{x}}_{1}-\mathrm{c}_{2} \dot{\mathrm{x}}_{2}+\left(\mathrm{k}_{1}+\mathrm{k}_{2}\right) \mathrm{x}_{1}-\mathrm{k}_{2} \mathrm{x}_{2}=0 \tag{A-8}
\end{align*}
$$

Assemble the equations in matrix form.

$$
\left[\begin{array}{cc}
\mathrm{m}_{1} & 0  \tag{A-9}\\
0 & \mathrm{~m}_{2}
\end{array}\right]\left[\begin{array}{l}
\ddot{\mathrm{x}}_{1} \\
\ddot{\mathrm{x}}_{2}
\end{array}\right]+\left[\begin{array}{cc}
\mathrm{c}_{1}+\mathrm{c}_{2} & -\mathrm{c}_{2} \\
-\mathrm{c}_{2} & \mathrm{c}_{2}+\mathrm{c}_{3}
\end{array}\right]\left[\begin{array}{c}
\dot{\mathrm{x}}_{1} \\
\dot{\mathrm{x}}_{2}
\end{array}\right]+\left[\begin{array}{cc}
\mathrm{k}_{1}+\mathrm{k}_{2} & -\mathrm{k}_{2} \\
-\mathrm{k}_{2} & \mathrm{k}_{2}+\mathrm{k}_{3}
\end{array}\right]\left[\begin{array}{l}
\mathrm{x}_{1} \\
\mathrm{x}_{2}
\end{array}\right]=\left[\begin{array}{l}
0 \\
0
\end{array}\right]
$$

The problem can now be solved as shown in the main text.

## APPENDIX B

## Example



$$
\begin{aligned}
\mathrm{m}_{1} & =2.0 \mathrm{lbm} \\
\mathrm{~m}_{2} & =1.0 \mathrm{lbm} \\
\mathrm{k}_{1} & =15,000 \mathrm{lbf} / \mathrm{in} \\
\mathrm{k}_{2} & =10,000 \mathrm{lbf} / \mathrm{in}
\end{aligned}
$$

Assume 5\% damping for each mode.
The initial displacement is

$$
\begin{array}{ll}
\mathrm{x}_{1} & =0.01 \mathrm{in} \\
\mathrm{x}_{2} & =0.02 \mathrm{in}
\end{array}
$$

The initial velocity is zero for each mass.

Assemble the equations in matrix form.

$$
\begin{align*}
& {\left[\begin{array}{cc}
\mathrm{m}_{1} & 0 \\
0 & \mathrm{~m}_{2}
\end{array}\right]\left[\begin{array}{l}
\ddot{\mathrm{x}}_{1} \\
\ddot{\mathrm{x}}_{2}
\end{array}\right]+\left[\begin{array}{cc}
\mathrm{k}_{1}+\mathrm{k}_{2} & -\mathrm{k}_{2} \\
-\mathrm{k}_{2} & \mathrm{k}_{2}
\end{array}\right]\left[\begin{array}{l}
\mathrm{x}_{1} \\
\mathrm{x}_{2}
\end{array}\right]=\left[\begin{array}{l}
0 \\
0
\end{array}\right]}  \tag{B-1}\\
& {\left[\begin{array}{cc}
2 / 386 & 0 \\
0 & 1 / 386
\end{array}\right]\left[\begin{array}{l}
\ddot{\mathrm{x}}_{1} \\
\ddot{\mathrm{x}}_{2}
\end{array}\right]+\left[\begin{array}{cc}
25,000 & -10,000 \\
-10,000 & 10,000
\end{array}\right]\left[\begin{array}{l}
\mathrm{x}_{1} \\
\mathrm{x}_{2}
\end{array}\right]=\left[\begin{array}{l}
0 \\
0
\end{array}\right]} \tag{B-2}
\end{align*}
$$

The results are shown in Figure 1.


Figure B-1.

The response was calculated using Matlab script: mdof_free.m

```
Natural Frequencies
    No. f(Hz)
1. 199.47
2. 424.51
```

