THE STEADY-STATE RESPONSE OF A TWO-DEGREE-OF-FREEDOM-SYSTEM TO A SINGLE HARMONIC FORCE

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Two-degree-of-freedom System

The method of generalized coordinates is demonstrated by an example. Consider the system in Figure 1.

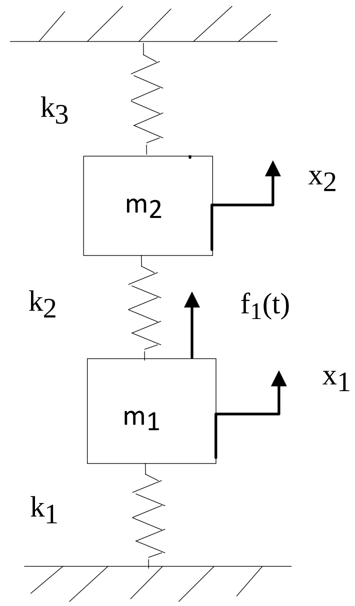

Figure 1.

A free-body diagram of mass 1 is given in Figure 2. A free-body diagram of mass 2 is given in Figure 3.

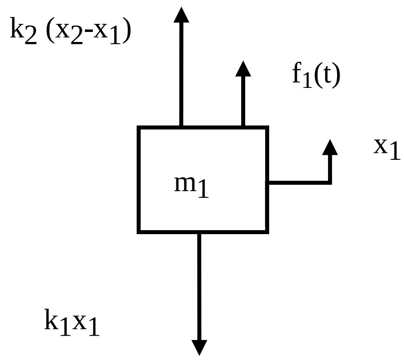

Figure 2.

Determine the equation of motion for mass 1.

$$\sum F = m_1 \ddot{x}_1 \tag{1}$$

$$m_1\ddot{x}_1 = f_1(t) + k_2(x_2 - x_1) - k_1 x_1 \tag{2}$$

$$m_1\ddot{x}_1 - k_2(x_2 - x_1) + k_1 x_1 = f_1(t) \tag{3}$$

$$m_1\ddot{x}_1 + k_2(-x_2 + x_1) + k_1 x_1 = f_1(t) \tag{4}$$

$$m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2 x_2 = f_1(t) \tag{5}$$

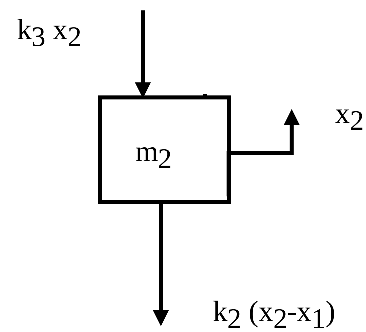

Figure 3.

Derive the equation of motion for mass 2.

$$\sum F = m_2 \ddot{x}_2 \tag{6}$$

$$m_2 \ddot{x}_2 = -k_2 (x_2 - x_1) - k_3 x_2 \tag{7}$$

$$m_2 \ddot{x}_2 + k_2 (x_2 - x_1) + k_3 x_2 = 0 \tag{8}$$

$$m_2 \ddot{x}_2 + (k_2 + k_3) x_2 - k_2 x_1 = 0 \tag{9}$$

Assemble the equations in matrix form.

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} f_1(t) \\ 0 \end{bmatrix} \tag{10}$$

<u>Decoupling</u>

Equation (10) is coupled via the stiffness matrix. An intermediate goal is to decouple the equation.

Simplify,

$$M\ddot{\overline{x}} + K\,\overline{x} = \overline{F} \tag{11}$$

where

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \tag{12}$$

$$K = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \tag{13}$$

$$\overline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{14}$$

$$\overline{F} = \begin{bmatrix} f_1(t) \\ 0 \end{bmatrix} \tag{15}$$

Consider the homogeneous form of equation (11).

$$M\ddot{\overline{x}} + K\,\overline{x} = \overline{0} \tag{16}$$

Seek a solution of the form

$$\overline{x} = \overline{q}\exp(j\omega t) \tag{17}$$

The q vector is the generalized coordinate vector.

Note that

$$\dot{\overline{x}} = j\omega\overline{q}\exp(j\omega t) \tag{18}$$

$$\ddot{\overline{x}} = -\omega^2\,\overline{q}\exp(j\omega t) \tag{19}$$

Substitute equations (17) through (19) into equation (16).

$$-\omega^2 M\bar{q}\exp(j\omega t) + K\bar{q}\exp(j\omega t) = \bar{0} \tag{20}$$

$$\left\{-\omega^2 M\bar{q} + K\bar{q}\right\}\exp(j\omega t) = \bar{0} \tag{21}$$

$$-\omega^2 M\bar{q} + K\bar{q} = \bar{0} \tag{22}$$

$$\left\{-\omega^2 M + K\right\}\bar{q} = \bar{0} \tag{23}$$

$$\left\{K - \omega^2 M\right\}\bar{q} = \bar{0} \tag{24}$$

Equation (24) is an example of a generalized eigenvalue problem. The eigenvalues can be found by setting the determinant equal to zero.

$$\det\left\{K - \omega^2 M\right\} = \bar{0} \tag{25}$$

$$\det\left\{\begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} - \omega^2\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\right\} = 0 \tag{26}$$

$$\det\left\{\begin{bmatrix} (k_1 + k_2) - \omega^2 m_1 & -k_2 \\ -k_2 & (k_2 + k_3) - \omega^2 m_2 \end{bmatrix}\right\} = 0 \tag{27}$$

$$\left[(k_1 + k_2) - \omega^2 m_1\right]\left[(k_2 + k_3) - \omega^2 m_2\right] - k_2^{\,2} = 0 \tag{28}$$

$$(k_1+k_2)(k_2+k_3)-\omega^2 m_1(k_2+k_3)-\omega^2 m_2(k_1+k_2)+\omega^4 m_1 m_2-k_2^2=0 \tag{29}$$

$$m_1 m_2\,\omega^4+[-m_1(k_2+k_3)-m_2(k_1+k_2)]\omega^2+k_1k_3+(k_1+k_3)k_2+k_2^2-k_2^2=0 \tag{30}$$

$$m_1 m_2\,\omega^4+[-m_1(k_2+k_3)-m_2(k_1+k_2)]\omega^2+k_1k_3+k_1k_2+k_2k_3=0 \tag{31}$$

The eigenvalues are the roots of the polynomial.

$$\omega_1^2=\frac{-b-\sqrt{b^2-4ac}}{2a} \tag{32}$$

$$\omega_2^2=\frac{-b+\sqrt{b^2-4ac}}{2a} \tag{33}$$

where

$$a=m_1m_2 \tag{34}$$

$$b=[-m_1(k_2+k_3)-m_2(k_1+k_2)] \tag{35}$$

$$c=k_1k_2+k_1k_3+k_2k_3 \tag{36}$$

The eigenvectors are found via the following equations.

$$\{K-\omega_1^2\,M\}\bar{q}_1=\bar{0} \tag{37}$$

$$\{K-\omega_2^2\,M\}\bar{q}_2=\bar{0} \tag{38}$$

where

$$\bar{q}_1 = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{39}$$

$$\bar{q}_2 = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \tag{40}$$

An eigenvector matrix Q can be formed. The eigenvectors are inserted in column format.

$$Q = \begin{bmatrix} \bar{q}_1 & \bar{q}_2 \end{bmatrix} \tag{41}$$

$$Q = \begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix} \tag{42}$$

The eigenvectors represent orthogonal mode shapes.

Each eigenvector can be multiplied by an arbitrary scale factor. A mass-normalized eigenvector matrix $\hat{Q}$ can be obtained such that the following orthogonality relations are obtained.

$$\hat{Q}^T M \hat{Q} = I \tag{43}$$

and

$$\hat{Q}^T K \hat{Q} = \Omega \tag{44}$$

where

I	is the identity matrix
Ω	is a diagonal matrix of eigenvalues

The superscript T represents transpose.

Note the mass-normalized forms

$$\hat{Q} = \begin{bmatrix} \hat{v}_1 & \hat{w}_1 \\ \hat{v}_2 & \hat{w}_2 \end{bmatrix} \tag{45}$$

$$\hat{Q}^T = \begin{bmatrix} \hat{v}_1 & \hat{v}_2 \\ \hat{w}_1 & \hat{w}_2 \end{bmatrix} \tag{46}$$

Rigorous proof of the orthogonality relationships is beyond the scope of this tutorial.

Further discussion is given in References 1 and 2.

Nevertheless, the orthogonality relationships are demonstrated by an example in this tutorial.

Now define a generalize coordinate $\eta(t)$ such that

$$\bar{x} = \hat{Q}\,\bar{\eta} \tag{47}$$

Substitute equation (47) into the equation of motion, equation (11).

$$M\hat{Q}\,\ddot{\bar{\eta}} + K\hat{Q}\,\bar{\eta} = \bar{F} \tag{48}$$

Premultiply by the transpose of the normalized eigenvector matrix.

$$\hat{Q}^T M\hat{Q}\,\ddot{\bar{\eta}} + \hat{Q}^T K\hat{Q}\,\bar{\eta} = \hat{Q}^T \bar{F} \tag{49}$$

The orthogonality relationships yield

$$I\,\ddot{\bar{\eta}} + \Omega\,\bar{\eta} = \hat{Q}^T \bar{F} \tag{50}$$

The equations of motion along with an added damping matrix become

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \ddot{\eta}_1 \\ \ddot{\eta}_2 \end{bmatrix} + \begin{bmatrix} 2\xi_1\omega_1 & 0 \\ 0 & 2\xi_2\omega_2 \end{bmatrix}\begin{bmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \end{bmatrix} + \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix}\begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} \hat{v}_1 & \hat{v}_2 \\ \hat{w}_1 & \hat{w}_2 \end{bmatrix}\begin{bmatrix} f_1(t) \\ 0 \end{bmatrix} \tag{51}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \ddot{\eta}_1 \\ \ddot{\eta}_2 \end{bmatrix} + \begin{bmatrix} 2\xi_1\omega_1 & 0 \\ 0 & 2\xi_2\omega_2 \end{bmatrix}\begin{bmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \end{bmatrix} + \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix}\begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} \hat{v}_1 \\ \hat{w}_1 \end{bmatrix} f_1(t) \tag{52}$$

Note that the two equations are decoupled in terms of the generalized coordinate.

Equation (51b) yields two equations

$$\ddot{\eta}_1 + 2\xi_1\omega_1\dot{\eta}_1 + \omega_1^2\eta_1 = \hat{v}_1 f_1(t) \tag{53}$$

$$\ddot{\eta}_2 + 2\xi_2\omega_2\dot{\eta}_2 + \omega_2^2\eta_2 = \hat{w}_1 f_1(t) \tag{54}$$

<u>Harmonic Force</u>

Now consider the special case of harmonic forcing functions

$$f_1(t) = B\exp(j\omega t) \tag{55}$$

Thus,

$$\ddot{\eta}_1 + 2\xi_1\omega_1\dot{\eta}_1 + \omega_1^2\eta_1 = \hat{v}_1 B\exp(j\omega t) \tag{56}$$

$$\ddot{\eta}_2 + 2\xi_2\omega_2\dot{\eta}_2 + \omega_2^2\eta_2 = \hat{w}_1 B\exp(j\omega t) \tag{57}$$

Assume response functions

$$\eta_1 = N_1 \exp(j\omega t) \tag{58}$$

$$\eta_2 = N_2 \exp(j\omega t) \tag{59}$$

$$\left\{-\omega^2 + j2\xi_1\omega\omega_1 + \omega_1^2\right\} N_1 \exp(j\omega t) = \hat{v}_1 B \exp(j\omega t) \tag{60}$$

$$\left\{-\omega^2 + j2\xi_2\omega\omega_2 + \omega_2^2\right\} N_2 \exp(j\omega t) = \hat{w}_1 B \exp(j\omega t) \tag{61}$$

$$\left\{\omega_1^2 - \omega^2 + j2\xi_1\omega\omega_1\right\} N_1 \exp(j\omega t) = \hat{v}_1 B \exp(j\omega t) \tag{62}$$

$$\left\{\omega_2^2 - \omega^2 + j2\xi_2\omega\omega_2\right\} N_2 \exp(j\omega t) = \hat{w}_1 B \exp(j\omega t) \tag{63}$$

Take the Fourier transform of each side.

$$\left\{\omega_1^2 - \omega^2 + j2\xi_1\omega\omega_1\right\} \hat{\eta}_1(\omega) = \hat{v}_1 \hat{B}(\omega) \tag{64}$$

$$\left\{\omega_2^2 - \omega^2 + j2\xi_2\omega\omega_2\right\} \hat{\eta}_2(\omega) = \hat{w}_1 \hat{B}(\omega) \tag{65}$$

$$\hat{\eta}_1(\omega) = \left\{\frac{\hat{v}_1}{\omega_1^2 - \omega^2 + j2\xi_1\omega\omega_1}\right\} \hat{B}(\omega) \tag{66}$$

$$\hat{\eta}_2(\omega) = \left\{\frac{\hat{w}_1}{\omega_2^2 - \omega^2 + j2\xi_2\omega\omega_2}\right\} \hat{B}(\omega) \tag{67}$$

The Fourier transforms of the physical displacements are found via

$$\overline{X} = \hat{Q}\,\overline{\eta} \tag{68}$$

$$\hat{Q} = \begin{bmatrix} \hat{v}_1 & \hat{w}_1 \\ \hat{v}_2 & \hat{w}_2 \end{bmatrix} \tag{69}$$

$$\overline{X} = \begin{bmatrix} \hat{v}_1 & \hat{w}_1 \\ \hat{v}_2 & \hat{w}_2 \end{bmatrix} \begin{bmatrix} \dfrac{\hat{v}_1}{\omega_1^2 - \omega^2 + j\,2\,\xi_1\,\omega\omega_1} \\[2ex] \dfrac{\hat{w}_1}{\omega_2^2 - \omega^2 + j\,2\,\xi_2\,\omega\omega_2} \end{bmatrix} \hat{B}(\omega) \tag{70}$$

$$X_1(\omega) = \left\{ \frac{\hat{v}_1^{\,2}}{\omega_1^2 - \omega^2 + j\,2\,\xi_1\,\omega\omega_1} + \frac{\hat{w}_1^{\,2}}{\omega_2^2 - \omega^2 + j\,2\,\xi_2\,\omega\omega_2} \right\} \hat{B}(\omega) \tag{71}$$

$$X_2(\omega) = \left\{ \frac{\hat{v}_1 \hat{v}_2}{\omega_1^2 - \omega^2 + j\,2\,\xi_1\,\omega\omega_1} + \frac{\hat{w}_1 \hat{w}_2}{\omega_2^2 - \omega^2 + j\,2\,\xi_2\,\omega\omega_2} \right\} \hat{B}(\omega) \tag{72}$$

<u>References</u>

1. Bathe, Finite Element Procedures in Engineering Analysis, Prentice-Hall, New Jersey, 1982. Section 12.3.1.
2. Weaver and Johnston, Structural Dynamics by Finite Elements, Prentice-Hall, New Jersey, 1987. Chapter 4.
3. T. Irvine, Table of Laplace Transforms, Vibrationdata, 2000.
4. T. Irvine, Partial Fraction Expansion, Rev F, Vibrationdata, 2010.

APPENDIX A

Example

Consider the system in Figure 1 with the values in Table 2.

Assume 5% damping for each mode. Assume zero initial conditions.

Table 1. Parameters		
Variable	Value	Unit
m_1	3.0	lbf sec^2/in
m_2	2.0	lbf sec^2/in
k_1	400,000	lbf/in
k_2	300,000	lbf/in
k_3	100,000	lbf/in
B_1	100	lbf
B_2	200	lbf
α	55	Hz
β	100	Hz

The mass matrix is

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \tag{A-1}$$

The stiffness matrix is

$$K = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} = \begin{bmatrix} 700{,}000 & -300{,}000 \\ -300{,}000 & 400{,}000 \end{bmatrix} \tag{A-2}$$

The analysis is performed using a Matlab script.

```
>> two_dof_force_steady
```

```
  two_dof_force_steady.m   ver 1.0  May 20, 2011

  by Tom Irvine   Email: tomirvine@aol.com

This program finds the eigenvalues and eigenvectors for a
two-degree-of-freedom system.
The equation of motion is:   M (d^2x/dt^2) + K x = 0

The program also find the transfer functions for a
harmonic force applied at mass 1


Enter units
1=English 2=metric
1
Enter mass unit
1=lbm  2=lbf sec^2/in
2
Assume symmetric mass and stiffness matrices.

Mass unit is lbf sec^2/in

Enter m11
3
Enter m12
0
Enter m22
2

Stiffness unit is lbf/in

Enter k11
700000
Enter k12
-300000
Enter k22
400000

The mass matrix is

m =

     3     0
     0     2


The stiffness matrix is

k =

      700000     -300000
```

```
     -300000      400000

 Natural Frequencies
 No.      f(Hz)
1.        48.552
2.        92.839

  Modes Shapes (column format)

ModeShapes =

    0.3797   -0.4349
    0.5326    0.4651


 Particpation Factors =

     2.204
   -0.3746

 Effective Modal Mass =

      4.86
    0.1403

 Total Modal Mass =       5.0000

 Enter the damping ratio for mode 1
 0.05

 Enter the damping ratio for mode 2
 0.05
```

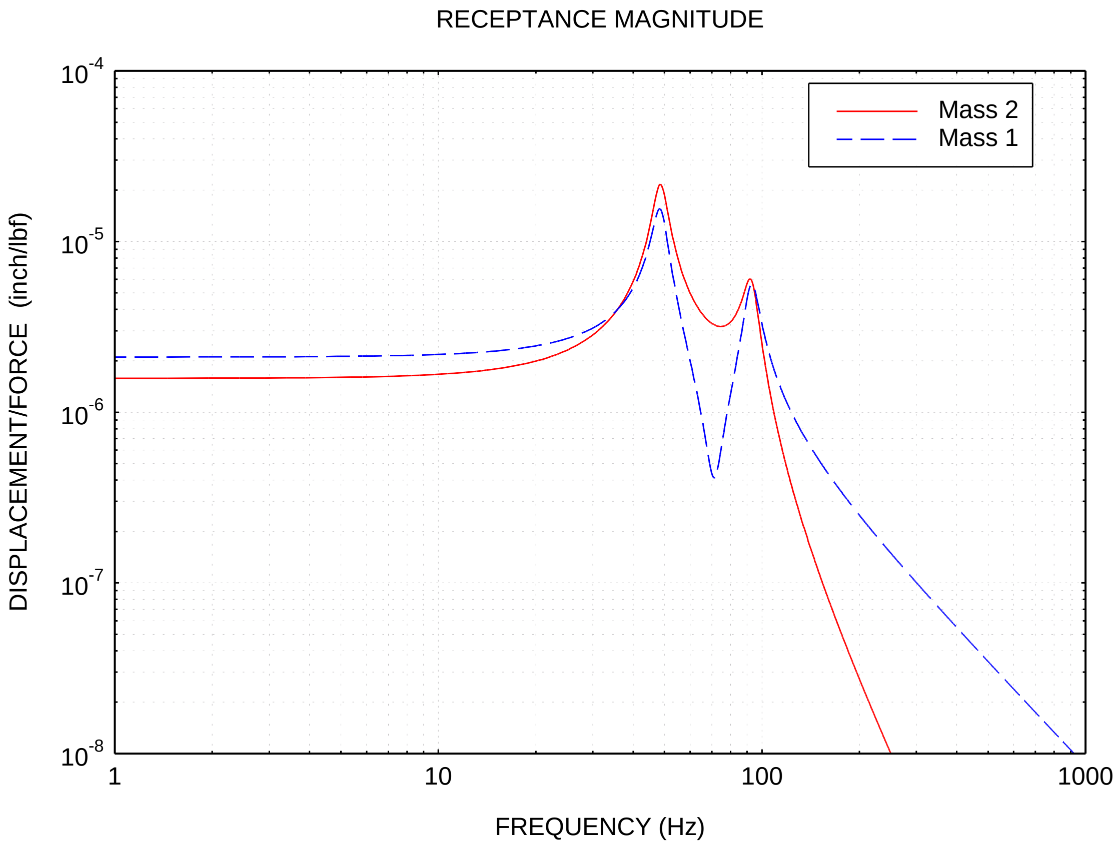

Figure A-1.

The curves represent the response of each mass to a unit harmonic force at mass 1.