# THE RESPONSE OF A TWO-DEGREE-OF-FREEDOM SYSTEM SUBJECTED TO WAVELET BASE EXCITATION

Revision C

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## Two-degree-of-freedom System, Modal Analysis

The method of generalized coordinates is demonstrated by an example. Consider the system in Figure 1.

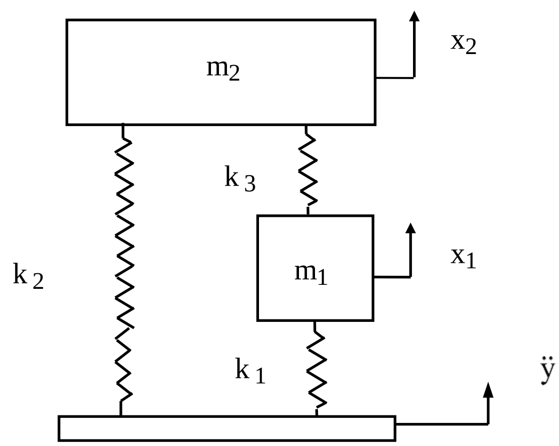


Figure 1.

The system also has damping, but it is modeled as modal damping.

A free-body diagram of mass 1 is given in Figure 2. A free-body diagram of mass 2 is given in Figure 3.

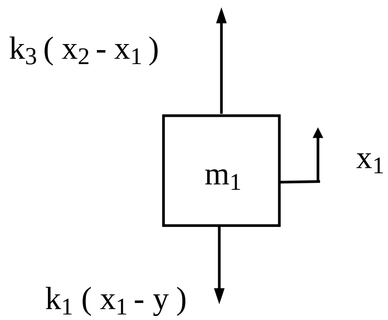


Figure 2.

Determine the equation of motion for mass 1.

$$\Sigma F = m_1 \ddot{x}_1 \tag{1}$$

$$m_1 \ddot{x}_1 = k_3(x_2 - x_1) - k_1(x_1 - y) \tag{2}$$

$$m_1 \ddot{x}_1 + k_1 x_1 - k_3(x_2 - x_1) = k_1 y \tag{3}$$

$$m_1 \ddot{x}_1 + k_1 x_1 + k_3(x_1 - x_2) = k_1 y \tag{4}$$

$$m_1 \ddot{x}_1 + (k_1 + k_3)x_1 - k_3 x_2 = k_1 y \tag{5}$$

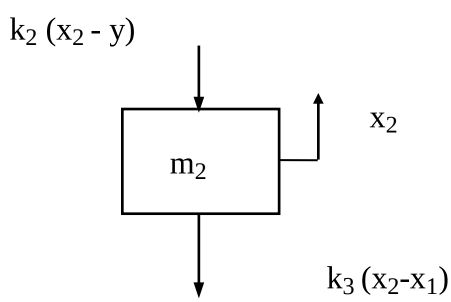


Figure 3.

Derive the equation of motion for mass 2.

$$\Sigma F = m_2 \ddot{x}_2 \tag{6}$$

$$m_2 \ddot{x}_2 = -k_3(x_2 - x_1) - k_2(x_2 - y) \tag{7}$$

$$m_2 \ddot{x}_2 + k_2 x_2 + k_3(x_2 - x_1) = k_2 y \tag{8}$$

$$m_2 \ddot{x}_2 + (k_2 + k_3)x_2 - k_3 x_1 = k_2 y \tag{9}$$

Assemble the equations in matrix form.

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_3 & -k_3 \\ -k_3 & k_2 + k_3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} k_1 y \\ k_2 y \end{bmatrix} \tag{10}$$

Define a relative displacement z such that

$$x_1 = z_1 + y \tag{11}$$

$$x_2 = z_2 + y \tag{12}$$

Substitute equations (11) and (12) into (10).

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{z}_1 + \ddot{y} \\ \ddot{z}_2 + \ddot{y} \end{bmatrix} + \begin{bmatrix} k_1 + k_3 & -k_3 \\ -k_3 & k_2 + k_3 \end{bmatrix}\begin{bmatrix} z_1 + y \\ z_2 + y \end{bmatrix} = \begin{bmatrix} k_1\ y \\ k_2\ y \end{bmatrix} \tag{13}$$

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{z}_1 \\ \ddot{z}_2 \end{bmatrix} + \begin{bmatrix} m_1\ddot{y} \\ m_2\ddot{y} \end{bmatrix} + \begin{bmatrix} k_1 + k_3 & -k_3 \\ -k_3 & k_2 + k_3 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_3 & -k_3 \\ -k_3 & k_2 + k_3 \end{bmatrix}\begin{bmatrix} y \\ y \end{bmatrix} = \begin{bmatrix} k_1 y \\ k_2 y \end{bmatrix} \tag{14}$$

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{z}_1 \\ \ddot{z}_2 \end{bmatrix} + \begin{bmatrix} m_1\ddot{y} \\ m_2\ddot{y} \end{bmatrix} + \begin{bmatrix} k_1 + k_3 & -k_3 \\ -k_3 & k_2 + k_3 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} k_1 y \\ k_2 y \end{bmatrix} = \begin{bmatrix} k_1\ y \\ k_2\ y \end{bmatrix} \tag{15}$$

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{z}_1 \\ \ddot{z}_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_3 & -k_3 \\ -k_3 & k_2 + k_3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} -m_1 \ddot{y} \\ -m_2 \ddot{y} \end{bmatrix} \tag{16}$$

Decoupling

Equation (16) is coupled via the stiffness matrix. An intermediate goal is to decouple the equation.

Simplify,

$$M \, \ddot{\bar{z}} + K \bar{z} = \bar{F} \tag{17}$$

where

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \tag{18}$$

$$K = \begin{bmatrix} k_1 + k_3 & -k_3 \\ -k_3 & k_2 + k_3 \end{bmatrix} \tag{19}$$

$$\bar{z} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \tag{20}$$

$$\bar{F} = \begin{bmatrix} -m_1 \ddot{y} \\ -m_2 \ddot{y} \end{bmatrix} \tag{21}$$

Consider the homogeneous form of equation (17).

$$M \, \ddot{\bar{z}} + K \bar{z} = \bar{0} \tag{22}$$

Seek a solution of the form

$$\bar{z} = \bar{q} \exp(j\omega t) \tag{23}$$

The q vector is the generalized coordinate vector.

Note that

$$\dot{\bar{z}} = j\omega\,\bar{q}\exp(j\omega t) \tag{24}$$

$$\ddot{\bar{z}} = -\omega^2\,\bar{q}\exp(j\omega t) \tag{25}$$

Substitute equations (23) through (25) into equation (22).

$$-\omega^2 M\,\bar{q}\exp(j\omega t) + K\bar{q}\exp(j\omega t) = \bar{0} \tag{26}$$

$$\left\{-\omega^2 M\,\bar{q} + K\bar{q}\right\}\exp(j\omega t) = \bar{0} \tag{27}$$

$$-\omega_n{}^2 M\,\bar{q} + K\bar{q} = \bar{0} \tag{28}$$

$$\left\{-\omega^2 M + K\right\}\bar{q} = \bar{0} \tag{29}$$

$$\left\{K - \omega^2 M\right\}\bar{q} = \bar{0} \tag{30}$$

Equation (30) is an example of a generalized eigenvalue problem. The eigenvalues can be found by setting the determinant equal to zero.

$$\det\left\{K - \omega^2 M\right\} = 0 \tag{31}$$

$$\det\left\{\begin{bmatrix} k_1 + k_3 & -k_3 \\ -k_3 & k_2 + k_3 \end{bmatrix} - \omega^2 \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\right\} = 0 \tag{32}$$

$$\det\begin{bmatrix} (k_1 + k_3) - \omega^2 m_1 & -k_3 \\ -k_3 & (k_2 + k_3) - \omega^2 m_2 \end{bmatrix} = 0 \tag{33}$$

$$\left[(k_1 + k_3) - \omega^2 m_1\right]\left[(k_2 + k_3) - \omega^2 m_2\right] - k_3{}^2 = 0 \tag{34}$$

$$\omega^4 m_1 m_2 - \omega^2\left[m_1(k_2 + k_3) + m_2(k_1 + k_3)\right] - k_3{}^2 = 0 \tag{35}$$

The eigenvalues are the roots of the polynomial.

$$\omega_1{}^2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \tag{36}$$

$$\omega_2{}^2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \tag{37}$$

where

$$a = m_1 m_2 \tag{38}$$

$$b = -\left[m_1\left(k_2 + k_3\right) + m_2\left(k_1 + k_3\right)\right] \tag{39}$$

$$c = -k_3{}^2 \tag{40}$$

The eigenvectors are found via the following equations.

$$\left\{K - \omega_1{}^2 M\right\}\bar{q}_1 = \bar{0} \tag{41}$$

$$\left\{K - \omega_2{}^2 M\right\}\bar{q}_2 = \bar{0} \tag{42}$$

where

$$\bar{q}_1 = \begin{bmatrix} q_{11} \\ q_{12} \end{bmatrix} \tag{34}$$

$$\bar{q}_2 = \begin{bmatrix} q_{21} \\ q_{22} \end{bmatrix} \tag{44}$$

An eigenvector matrix Q can be formed. The eigenvectors are inserted in column format.

$$Q = \left[\ \bar{q}_1 \quad | \quad \bar{q}_2\ \right] \tag{45}$$

$$Q = \begin{bmatrix} q_{11} & q_{21} \\ q_{12} & q_{22} \end{bmatrix} \tag{46}$$

The eigenvectors represent orthogonal mode shapes.

Each eigenvector can be multiplied by an arbitrary scale factor. A mass-normalized eigenvector matrix $\hat{Q}$ can be obtained such that the following orthogonality relations are obtained.

$$\hat{Q}^T M \hat{Q} = I \tag{47}$$

and

$$\hat{Q}^T K \hat{Q} = \Omega \tag{48}$$

where

superscript T represents transpose

I is the identity matrix

$\Omega$ is a diagonal matrix of eigenvalues

Note that

$$\hat{Q} = \begin{bmatrix} \hat{q}_{11} & \hat{q}_{12} \\ \hat{q}_{21} & \hat{q}_{22} \end{bmatrix} \tag{49a}$$

$$\hat{Q}^T = \begin{bmatrix} \hat{q}_{11} & \hat{q}_{21} \\ \hat{q}_{12} & \hat{q}_{22} \end{bmatrix} \tag{49b}$$

Rigorous proof of the orthogonality relationships is beyond the scope of this tutorial. Further discussion is given in the references.

Nevertheless, the orthogonality relationships are demonstrated by an example in this tutorial.

Now define a modal coordinate $\eta(t)$ such that

$$\bar{z} = \hat{Q}\ \bar{\eta} \tag{50a}$$

$$z_1 = \hat{q}_{11}\,\eta_1 + \hat{q}_{12}\,\eta_2 \tag{50b}$$

$$z_2 = \hat{q}_{21}\,\eta_1 + \hat{q}_{22}\,\eta_2 \tag{50c}$$

Recall

$$x_1 = z_1 + y \tag{51a}$$

$$x_2 = z_2 + y \tag{51b}$$

The displacement terms are

$$x_1 = y + \hat{q}_{11}\,\eta_1 + \hat{q}_{12}\,\eta_2 \tag{51a}$$

$$x_2 = y + \hat{q}_{21}\,\eta_1 + \hat{q}_{22}\,\eta_2 \tag{52b}$$

The velocity terms are

$$\dot{x}_1 = \dot{y} + \hat{q}_{11}\,\dot{\eta}_1 + \hat{q}_{12}\,\dot{\eta}_2 \tag{53a}$$

$$\dot{x}_2 = \dot{y} + \hat{q}_{21}\,\dot{\eta}_1 + \hat{q}_{22}\,\dot{\eta}_2 \tag{53b}$$

The acceleration terms are

$$\ddot{x}_1 = \ddot{y} + \hat{q}_{11}\,\ddot{\eta}_1 + \hat{q}_{12}\,\ddot{\eta}_2 \tag{54a}$$

$$\ddot{x}_2 = \ddot{y} + \hat{q}_{21}\,\ddot{\eta}_1 + \hat{q}_{22}\,\ddot{\eta}_2 \tag{54b}$$

Substitute equation (50a) into the equation of motion, equation (17).

$$M\hat{Q}\,\overline{\ddot{\eta}} + K\hat{Q}\,\overline{\eta} = \overline{F} \tag{55}$$

Premultiply by the transpose of the normalized eigenvector matrix.

$$\hat{Q}^T M\hat{Q}\,\overline{\ddot{\eta}} + \hat{Q}^T K\hat{Q}\,\overline{\eta} = \hat{Q}^T\overline{F} \tag{56}$$

The orthogonality relationships yield

$$I\,\overline{\ddot{\eta}} + \Omega\,\overline{\eta} = \hat{Q}^T\overline{F} \tag{57}$$

For the sample problem, equation (57) becomes

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \ddot{\eta}_1 \\ \ddot{\eta}_2 \end{bmatrix} + \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix}\begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} \hat{q}_{11} & \hat{q}_{21} \\ \hat{q}_{12} & \hat{q}_{22} \end{bmatrix}\begin{bmatrix} -m_1\ddot{y} \\ -m_2\ddot{y} \end{bmatrix} \tag{58}$$

Note that the two equations are decoupled in terms of the modal coordinate.

Now assume modal damping by adding an uncoupled damping matrix.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \ddot{\eta}_1 \\ \ddot{\eta}_2 \end{bmatrix} + \begin{bmatrix} 2\xi_1\omega_1 & 0 \\ 0 & 2\xi_2\omega_2 \end{bmatrix}\begin{bmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \end{bmatrix} + \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix}\begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} \hat{q}_{11} & \hat{q}_{21} \\ \hat{q}_{12} & \hat{q}_{22} \end{bmatrix}\begin{bmatrix} -m_1\ddot{y} \\ -m_2\ddot{y} \end{bmatrix} \tag{59}$$

Now consider the initial conditions. Recall

$$\bar{z} = \hat{Q}\ \bar{\eta} \tag{60}$$

Thus,

$$\bar{z}(0) = \hat{Q}\ \bar{\eta}(0) \tag{61}$$

Premultiply by $\hat{Q}^T\ M$.

$$\hat{Q}^T\ M\,\bar{z}(0) = \hat{Q}^T\ M\,\hat{Q}\,\eta(0) \tag{62}$$

Recall

$$\hat{Q}^T M\,\hat{Q} = I \tag{63}$$

$$\hat{Q}^T\ M\,\bar{z}(0) = I\ \ \eta(0) \tag{64}$$

$$\hat{Q}^T\ M\,\bar{z}(0) = \ \eta(0) \tag{65}$$

Finally, the transformed initial displacement is

$$\eta(0) = \hat{Q}^T\ M\,\bar{z}(0) \tag{66}$$

Similarly, the transformed initial velocity is

$$\dot{\eta}(0) = \hat{Q}^T\ M\,\dot{\bar{z}}(0) \tag{67}$$

The product of the first two matrices on the left side of equation (59) equals a vector of participation factors.

$$\begin{bmatrix} -\Gamma_1 \\ -\Gamma_2 \end{bmatrix} = \begin{bmatrix} \hat{q}_{11} & \hat{q}_{21} \\ \hat{q}_{12} & \hat{q}_{22} \end{bmatrix} \begin{bmatrix} m_{11} \\ m_{22} \end{bmatrix} \tag{68}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{\eta}_1 \\ \ddot{\eta}_2 \end{bmatrix} + \begin{bmatrix} 2\xi_1\omega_1 & 0 \\ 0 & 2\xi_2\omega_2 \end{bmatrix} \begin{bmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \end{bmatrix} + \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} -\Gamma_1 \ddot{y} \\ -\Gamma_2 \ddot{y} \end{bmatrix} \tag{69}$$

Equation (69) can be solved in terms of Laplace transforms.

<u>Wavelet Excitation</u>

The base excitation function is:

$$\ddot{y}(t) = \begin{cases} A \sin\left[\dfrac{2\pi f t}{N}\right] \sin\left[2\pi f t\right], & 0 \le t \le T \\ 0, & t > T \end{cases} \tag{70}$$

where

| | | |
|---|---|---|
| A | = | wavelet acceleration amplitude |
| f | = | wavelet frequency |
| N | = | number of half-sines, odd integer $\geq$ 3 |
| T | = | N / (2 f) |

The base excitation may also be expressed as:

$$\ddot{y}(t) = \begin{cases} -\frac{A}{2}\cos\left[(N+1)\frac{2\pi\, f\, t}{N}\right] + \frac{A}{2}\cos\left[(N-1)\frac{2\pi\, f\, t}{N}\right], & 0 \le t \le T \\ \\ 0\,, \quad t > T \end{cases}$$

(71)

The equation of motion becomes

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \ddot{\eta}_1 \\ \ddot{\eta}_2 \end{bmatrix} + \begin{bmatrix} 2\xi_1\,\omega_1 & 0 \\ 0 & 2\xi_2\,\omega_2 \end{bmatrix}\begin{bmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \end{bmatrix} + \begin{bmatrix} {\omega_1}^2 & 0 \\ 0 & {\omega_2}^2 \end{bmatrix}\begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} -\Gamma_1\,\ddot{y} \\ -\Gamma_2\,\ddot{y} \end{bmatrix}$$

(72)

The equation of motion for mode i is:

$$\ddot{\eta}_i + 2\xi\omega_n\dot{\eta}_i + {\omega_n}^2\eta_i = \frac{A\Gamma_i}{2}\cos\left[(N+1)\frac{2\pi\, f\, t}{N}\right] - \frac{A\Gamma_i}{2}\cos\left[(N-1)\frac{2\pi\, f\, t}{N}\right], \qquad 0 \le t \le T$$

(73)

Let

$$\alpha = (N+1)\frac{2\pi f}{N} \tag{74}$$

$$\beta = (N-1)\frac{2\pi f}{N} \tag{75}$$

$$B = A\;\Gamma_i\;/\;2 \tag{76}$$

By substitution,

$$\ddot{\eta}_i + 2\xi\omega_n\dot{\eta}_i + \omega_n{}^2\eta_i = B\cos(\alpha t) - B\cos(\beta t) \quad , \quad 0 \le t \le T \tag{77}$$

The solution to equation (77) is given in Reference 1. Excerpts are given as follows.

Define the following coefficients. Note that there is a set of coefficients for each mode. The subscript i is omitted for brevity.

$$C_1 = B\frac{-\left(\alpha^2 - \omega_n^2\right)}{\left[\left(\alpha^2 - \omega_n^2\right)^2 + (2\xi\alpha\,\omega_n)^2\right]} \tag{78}$$

$$C_2 = B\frac{2\xi\alpha^2\omega_n}{\left[\left(\alpha^2 - \omega_n^2\right)^2 + (2\xi\alpha\,\omega_n)^2\right]} \tag{79}$$

$$C_3 = B\frac{\left(\alpha^2 - \omega_n^2\right)}{\left[\left(\alpha^2 - \omega_n^2\right)^2 + (2\xi\alpha\,\omega_n)^2\right]} \tag{80}$$

$$C_4 = B\frac{-2\xi\omega_n^3}{\left[\left(\alpha^2 - \omega_n^2\right)^2 + (2\xi\alpha\,\omega_n)^2\right]} \tag{81}$$

$$C_5 = -B \frac{-\left(\beta^2 - \omega_n^2\right)}{\left[\left(\beta^2 - \omega_n^2\right)^2 + \left(2\xi\beta\omega_n\right)^2\right]} \tag{82}$$

$$C_6 = -B \frac{2\xi\beta^2\omega_n}{\left[\left(\beta^2 - \omega_n^2\right)^2 + \left(2\xi\beta\omega_n\right)^2\right]} \tag{83}$$

$$C_7 = -B \frac{\left(\beta^2 - \omega_n^2\right)}{\left[\left(\beta^2 - \omega_n^2\right)^2 + \left(2\xi\beta\omega_n\right)^2\right]} \tag{84}$$

$$C_8 = -B \frac{-2\xi\omega_n^3}{\left[\left(\beta^2 - \omega_n^2\right)^2 + \left(2\xi\beta\omega_n\right)^2\right]} \tag{85}$$

$$C_{10} = C_3 + C_7 \tag{86}$$

$$C_{11} = C_4 + C_8 \tag{87}$$

$$C_{20} = C_{11} - \xi\omega_n C_{10} \tag{88}$$

The total modal relative displacement for $0 \le t \le T$ is

$$\eta_i(t) = \exp(-\xi\omega_n t)\left\{\eta(0)\cos(\omega_d t) + \left\{\frac{\dot{\eta}_i(0) + (\xi\omega_n)\eta(0)}{\omega_d}\right\}\sin(\omega_d t)\right\}$$

$$+ \ C_1 \cos(\alpha t) \ + \ \frac{C_2}{\alpha}\sin(\alpha t) \ + \ C_5 \cos(\beta t) \ + \ \frac{C_6}{\beta}\sin(\beta t)$$

$$+ \ \exp(-\xi\omega_n t)\left[ C_{10} \cos(\omega_d t) + \frac{1}{\omega_d} C_{20} \sin(\omega_d t)\right]$$

(89)

The relative displacement for $t > T$ is found by adding a delay into equation (89).

$$\eta_i(t) = \exp(-\xi\omega_n(t-T))\left\{\eta_i(T)\cos(\omega_d(t-T)) + \left\{\frac{\dot{\eta}_i(T) + (\xi\omega_n)\eta_i(T)}{\omega_d}\right\}\sin(\omega_d(t-T))\right\}$$

(90)

Let

$$R_3(t) = \exp(-\xi\omega_n t)\left[ C_{10} \cos(\omega_d t) + \frac{1}{\omega_d} C_{20} \sin(\omega_d t)\right] \quad (91)$$

The total modal relative velocity for $0 \leq t \leq T$ is

$$\dot{\eta}_i(t) = \exp\left(-\xi\omega_n t\right)\left\{\dot{\eta}_i(0)\cos\left(\omega_d t\right) + \frac{\omega_n}{\omega_d}\left\{-\omega_n \eta_i(0) - \xi\,\dot{\eta}_i(0)\right\}\sin\left(\omega_d t\right)\right\}$$

$$- \ \alpha C_1 \sin(\alpha t) \ + \ C_2 \cos(\alpha t) \ - \ \beta C_5 \sin(\beta t) \ + \ C_6 \cos(\beta t)$$

$$-\xi\omega_n R_3(t)$$

$$+\exp(-\xi\omega_n t)\left[-\omega_d C_{10}\sin(\omega_d t) + C_{20}\cos(\omega_d t)\right] \tag{92}$$

$$\dot{\eta}_i(t) = \exp\left(-\xi\omega_n\left(t-T\right)\right)\left\{\dot{\eta}_i(T)\cos\left(\omega_d\left(t-T\right)\right) + \frac{\omega_n}{\omega_d}\left\{-\omega_n \eta(T) - \xi\,\dot{\eta}_i(T)\right\}\sin\left(\omega_d\left(t-T\right)\right)\right\} \tag{93}$$

The relative acceleration can then be found from

$$\ddot{\eta}_i = -2\xi\omega_n\dot{\eta}_i - \omega_n{}^2\eta_i + B\cos(\alpha t) - B\cos(\beta t) \quad , \quad 0 \leq t \leq T \tag{94}$$

$$\ddot{\eta}_i = -2\xi\omega_n\dot{\eta}_i - \omega_n{}^2\eta_i \quad , \quad t > 0 \tag{95}$$

The physical relative displacement is then calculated per

$$\bar{z} = \hat{Q}\ \bar{\eta} \tag{96}$$

The physical relative velocity and relative acceleration terms can then be calculated using the appropriate derivatives.

The physical absolute acceleration is then

$$\ddot{x}_i = \ddot{z}_i + \ddot{y} \tag{97}$$

References


1. T. Irvine, The Generalized Coordinate Method for Discrete Systems, Revision D, Vibrationdata, 2010.

2. T. Irvine, Response of a Single-degree-of-freedom System Subjected to a Classical Pulse Base Excitation, Vibrationdata, 1999.

3. T. Irvine, The Response of a Single-degree-of-freedom System Subjected to a Wavelet Pulse Base Excitation, Vibrationdata, 2008.

4. T. Irvine, Effective Modal Mass & Modal Participation Factors, Revision E, Vibrationdata, 2010.

APPENDIX A

EXAMPLE

Normal Modes Analysis

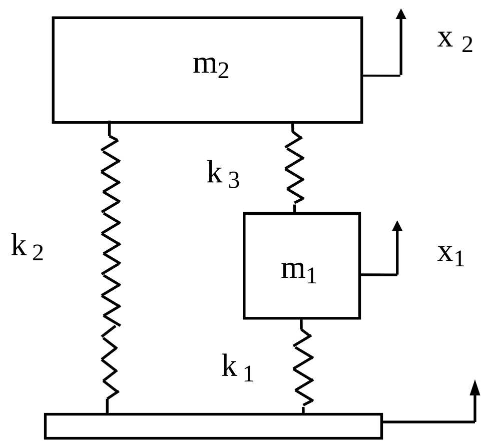


Figure A-1.

Consider the system in Figure A-1. Assign the values in Table A-1.

| Table A-1. Parameters | | |
|---|---|---|
| Variable | Value | Unit |
| $m_1$ | 3.0 | lbf sec^2/in |
| $m_2$ | 2.0 | lbf sec^2/in |
| $k_1$ | 400,000 | lbf/in |
| $k_2$ | 300,000 | lbf/in |
| $k_3$ | 100,000 | lbf/in |

Furthermore, assume

1. Each mode has a damping value of 5%.
2. Zero initial conditions

Next, assume that the base input function is a 10 G, 10 msec half-sine pulse.

Solve for the acceleration response time histories. The homogeneous, undamped problem is

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{z}_1 \\ \ddot{z}_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_3 & -k_3 \\ -k_3 & k_2 + k_3 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} -m_1\ddot{y} \\ -m_2\ddot{y} \end{bmatrix} \tag{A-1}$$

$$\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} \ddot{z}_1 \\ \ddot{z}_2 \end{bmatrix} + \begin{bmatrix} 500{,}000 & -100{,}000 \\ -100{,}000 & 400{,}000 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{A-2}$$

The eigenvalue problem is

$$\begin{bmatrix} 500{,}000 - 2\omega^2 & -100{,}000 \\ -100{,}000 & 400{,}000 - \omega^2 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{A-3}$$

The analysis is performed using Matlab script: twodof_wavelet.m.

The results are:

```
mass =

     3     0
     0     2


stiff =

      500000     -100000
     -100000      400000

 Natural Frequencies
 No.       f(Hz)

1.        59.388
2.          75.9


  Modes Shapes (column format)

ModeShapes =

    0.4792   -0.3220
    0.3943    0.5869
```

```
Enter the damping ratio for mode 1 0.05
Enter the damping ratio for mode 2 0.05

Particpation Factors =

    2.226
   0.2079

Enter the wavelet amplitude (G) 1
Enter wavelet frequency (Hz) 75
Enter number of half-sines  11


dof 1
maximum acceleration =   2.47 G
minimum acceleration =  -2.42 G

dof 2
maximum acceleration =   2.28 G
minimum acceleration =  -2.19 G
```

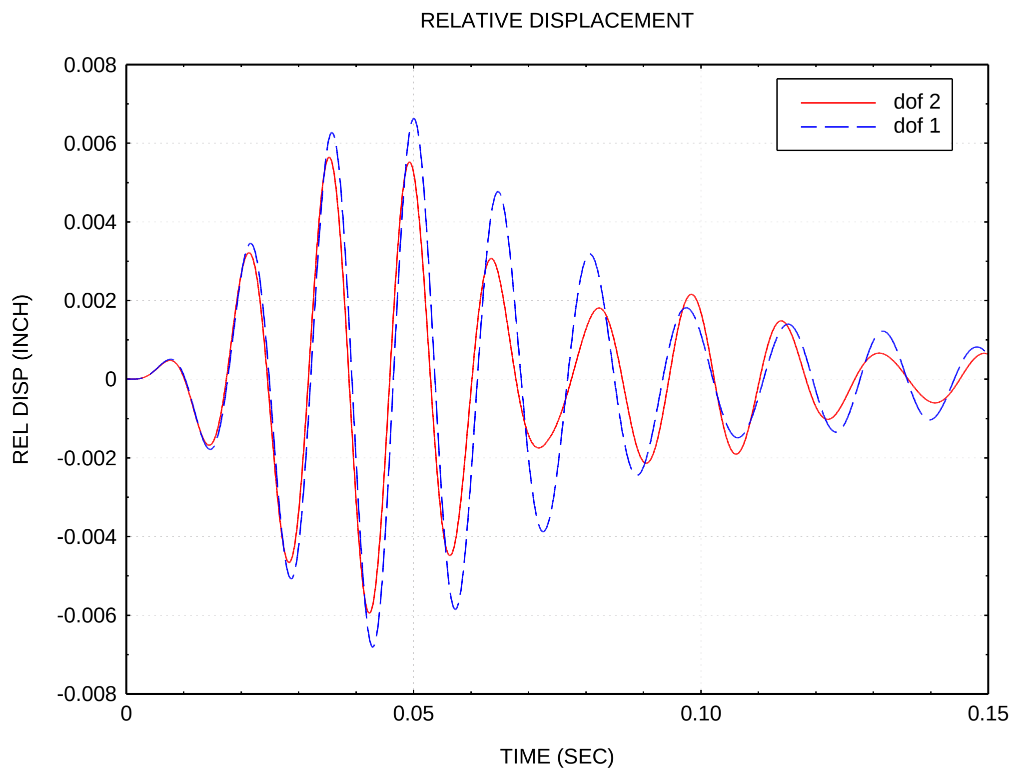
RELATIVE DISPLACEMENT
REL DISP (INCH)
0.008
0.006
0.004
0.002
0
-0.002
-0.004
-0.006
-0.008
0
0.05
0.10
0.15
TIME (SEC)
dof 2
dof 1

Figure A-2.

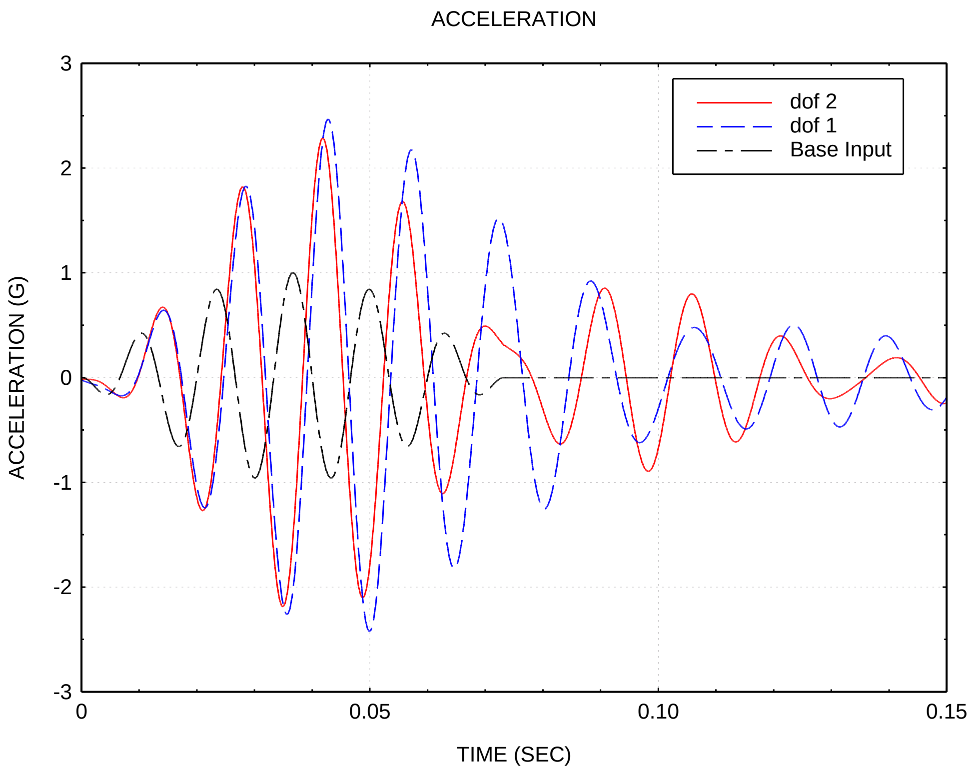
ACCELERATION
dof 2
dof 1
Base Input
ACCELERATION (G)
TIME (SEC)
3
2
1
0
-1
-2
-3
0
0.05
0.10
0.15

Figure A-3.

APPENDIX B

Modal Participation Factor

Consider a discrete dynamic system governed by the following equation

$$M\ddot{\bar{x}} + K\bar{x} = \bar{F} \tag{B-1}$$

where

- $M$ is the mass matrix
- $K$ is the stiffness matrix
- $\ddot{\bar{x}}$ is the acceleration vector
- $\bar{x}$ is the displacement vector
- $\bar{F}$ is the forcing function or base excitation function

A solution to the homogeneous form of equation (B-1) can be found in terms of eigenvalues and eigenvectors. The eigenvectors represent vibration modes.

Let $\phi$ be the eigenvector matrix.

The system's generalized mass matrix $\hat{m}$ is given by

$$\hat{m} = \phi^{T} M \phi \tag{B-2}$$

Let $\bar{r}$ be the influence vector which represents the displacements of the masses resulting from static application of a unit ground displacement.

Define a coefficient vector $\bar{L}$ as

$$\bar{L} = \phi^{T} M \, \bar{r} \tag{B-3}$$

The modal participation factor matrix $\Gamma_{i}$ for mode i is

$$\Gamma_{i} = \frac{\bar{L}_{i}}{\hat{m}_{ii}} \tag{B-4}$$

The effective modal mass $m_{eff,i}$ for mode i is

$$m_{eff,i} = \frac{\bar{L}_i^{\;2}}{\hat{m}_{ii}} \tag{B-5}$$