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Revision E

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Two-degree-of-freedom System

Consider the system in Figure 1.

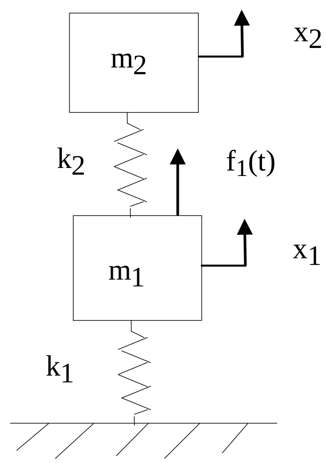

Figure 1.

A free-body diagram of mass 1 is given in Figure 2. A free-body diagram of mass 2 is given in Figure 3.

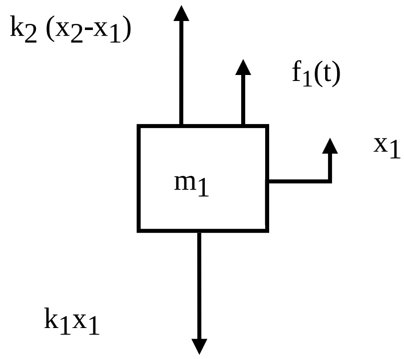

Figure 2.

Determine the equation of motion for mass 1.

$$\sum F = m_1 \ddot{x}_1 \tag{1}$$

$$m_1\ddot{x}_1 = f_1(t) + k_2(x_2 - x_1) - k_1x_1 \tag{2}$$

$$m_1\ddot{x}_1 - k_2(x_2 - x_1) + k_1x_1 = f_1(t) \tag{3}$$

$$m_1\ddot{x}_1 + k_2(-x_2 + x_1) + k_1x_1 = f_1(t) \tag{4}$$

$$m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 = f_1(t) \tag{5}$$

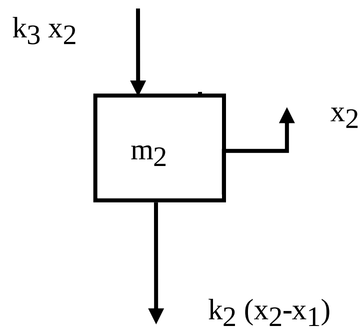

Figure 3.

Derive the equation of motion for mass 2.

$$\sum F = m_2 \, \ddot{x}_2 \tag{6}$$

$$m_2 \ddot{x}_2 = -k_2 (x_2 - x_1) \tag{7}$$

$$m_2 \ddot{x}_2 + k_2 (x_2 - x_1) = 0 \tag{8}$$

$$m_2 \ddot{x}_2 + k_2 x_2 - k_2 x_1 = 0 \tag{9}$$

Assemble the equations in matrix form.

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} f_1(t) \\ 0 \end{bmatrix} \tag{10}$$

The homogenous form is

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{11}$$

Seek a solution of the form

$$\bar{x} = \bar{q} \exp(j\omega t) \tag{12}$$

The q vector is the generalized coordinate vector.

Note that

$$\bar{\dot{x}} = j\omega \bar{q} \exp(j\omega t) \tag{13}$$

$$\bar{\ddot{x}} = -\omega^2 \bar{q} \exp(j\omega t) \tag{14}$$

The generalized eigenvalue problem is

$$\left\{ \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} - \omega^2 \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \right\} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \exp(j\omega t) = 0 \tag{15}$$

$$\det \left\{ \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} - \omega^2 \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \right\} = 0 \tag{16}$$

$$\det \begin{bmatrix} k_1 + k_2 - \omega^2 m_1 & -k_2 \\ -k_2 & k_2 - \omega^2 m_2 \end{bmatrix} = 0 \tag{17}$$

$$\left(k_1 + k_2 - \omega^2 m_1\right)\left(k_2 - \omega^2 m_2\right) - {k_2}^2 = 0 \tag{18}$$

$$k_1 k_2 + {k_2}^2 - \omega^2 m_1 k_2 - \omega^2 m_2 k_1 - \omega^2 m_2 k_2 + \omega^4 m_1 m_2 = 0 \tag{19}$$

$$\omega^4 m_1 m_2 - \omega^2 \left(m_1 k_2 + m_2 k_1 + m_2 k_2\right) + k_1 k_2 = 0 \tag{20}$$

$$\omega^4 m_1 m_2 - \omega^2 [m_1 k_2 + m_2 (k_1 + k_2)] + k_1 k_2 = 0 \tag{21}$$

The eigenvalues are the roots of the polynomial.

$$\omega_1^2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \tag{22}$$

$$\omega_2^2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \tag{23}$$

where

$$a = m_1 m_2 \tag{24}$$

$$b = -[m_1 k_2 + m_2 (k_1 + k_2)] \tag{25}$$

$$c = k_1 k_2 \tag{26}$$

<u>Frequency Response Function</u>

Let

$$f_1(t) = F_1 \exp(j\omega t) \tag{27}$$

$$x_1(t) = X_1 \exp(j\omega t) \tag{28}$$

$$x_2(t) = X_2 \exp(j\omega t) \tag{29}$$

$$-\omega^2 m_1 X_1 \exp(j\omega t) + (k_1 + k_2) X_1 \exp(j\omega t) - k_2 X_2 \exp(j\omega t) = F_1 \exp(j\omega t) \quad (30)$$

$$-\omega^2 m_1 X_1 + (k_1 + k_2) X_1 - k_2 X_2 = F_1 \quad (31)$$

$$\left[(k_1 + k_2) - \omega^2 m_1\right] X_1 - k_2 X_2 = F_1 \quad (32)$$

Recall the equation for mass 2.

$$m_2 \ddot{x}_2 + k_2 x_2 - k_2 x_1 = 0 \quad (33)$$

$$-\omega^2 m_2 X_2 + k_2 X_2 - k_2 X_1 = 0 \quad (34)$$

$$\left[k_2 - \omega^2 m_2\right] X_2 - k_2 X_1 = 0 \quad (35)$$

$$X_2 = \frac{k_2}{\left[k_2 - \omega^2 m_2\right]} X_1 \quad (36)$$

By substitution,

$$\left[(k_1 + k_2) - \omega^2 m_1\right] X_1 - \frac{k_2^{\,2}}{\left[k_2 - \omega^2 m_2\right]} X_1 = F_1 \quad (37)$$

$$\left[(k_1 + k_2) - \omega^2 m_1\right]\left[k_2 - \omega^2 m_2\right] X_1 - k_2^{\,2} X_1 = F_1 \left[k_2 - \omega^2 m_2\right] \quad (38)$$

$$\left\{\left[(k_1 + k_2) - \omega^2 m_1\right]\left[k_2 - \omega^2 m_2\right] - k_2^{\,2}\right\} X_1 = F_1 \left[k_2 - \omega^2 m_2\right] \quad (39)$$

$$X_1 = \frac{F_1\left[k_2 - \omega^2 m_2\right]}{\left[(k_1 + k_2) - \omega^2 m_1\right]\left[k_2 - \omega^2 m_2\right] - k_2^2} \tag{40}$$

$$\frac{X_1}{F_1} = \frac{k_2 - \omega^2 m_2}{\left[(k_1 + k_2) - \omega^2 m_1\right]\left[k_2 - \omega^2 m_2\right] - k_2^2} \tag{41}$$

$$\frac{X_1}{F_1} = \frac{k_2 - \omega^2 m_2}{k_1\left\{\left[1 + \frac{k_2}{k_1} - \omega^2 \frac{m_1}{k_1}\right]\left[k_2 - \omega^2 m_2\right] - \frac{k_2^2}{k_1}\right\}} \tag{42}$$

$$\frac{k_1 X_1}{F_1} = \frac{k_2 - \omega^2 m_2}{\left\{\left[1 + \frac{k_2}{k_1} - \omega^2 \frac{m_1}{k_1}\right]\left[k_2 - \omega^2 m_2\right] - \frac{k_2^2}{k_1}\right\}} \tag{43}$$

$$\frac{k_1 X_1}{F_1} = \frac{k_2 - \omega^2 m_2}{k_2\left\{\left[1 + \frac{k_2}{k_1} - \omega^2 \frac{m_1}{k_1}\right]\left[1 - \omega^2 \frac{m_2}{k_2}\right] - \frac{k_2}{k_1}\right\}} \tag{44}$$

$$\frac{k_1 X_1}{F_1} = \frac{1 - \omega^2 \frac{m_2}{k_2}}{\left\{\left[1 + \frac{k_2}{k_1} - \omega^2 \frac{m_1}{k_1}\right]\left[1 - \omega^2 \frac{m_2}{k_2}\right] - \frac{k_2}{k_1}\right\}} \tag{45}$$

Let

$$\omega_{11}^{2} = \frac{k_1}{m_1} \tag{46}$$

$$\omega_{22}^{2} = \frac{k_2}{m_2} \tag{47}$$

By substitution,

$$\frac{k_1 X_1}{F_1} = \frac{1-\left(\frac{\omega}{\omega_{22}}\right)^2}{\left[1+\frac{k_2}{k_1}-\left(\frac{\omega}{\omega_{11}}\right)^2\right]\left[1-\left(\frac{\omega}{\omega_{22}}\right)^2\right]-\frac{k_2}{k_1}} \tag{48}$$

$$X_2 = \left[\frac{k_2}{k_2-\omega^2 m_2}\right] X_1 \tag{49}$$

$$X_1 = \frac{F_1}{k_1}\left\{\frac{1-\left(\frac{\omega}{\omega_{22}}\right)^2}{\left[1+\frac{k_2}{k_1}-\left(\frac{\omega}{\omega_{11}}\right)^2\right]\left[1-\left(\frac{\omega}{\omega_{22}}\right)^2\right]-\frac{k_2}{k_1}}\right\} \tag{50}$$

$$X_2 = \frac{F_1}{\left[k_2 - \omega^2 m_2\right]} \frac{k_2}{k_1} \left\{ \frac{1 - \left(\frac{\omega}{\omega_{22}}\right)^2}{\left[1 + \frac{k_2}{k_1} - \left(\frac{\omega}{\omega_{11}}\right)^2\right]\left[1 - \left(\frac{\omega}{\omega_{22}}\right)^2\right] - \frac{k_2}{k_1}} \right\} \tag{51}$$

$$X_2 = \frac{F_1}{\left[\frac{k_2}{m_2} - \omega^2\right]} \frac{k_2}{m_2 k_1} \left\{ \frac{1 - \left(\frac{\omega}{\omega_{22}}\right)^2}{\left[1 + \frac{k_2}{k_1} - \left(\frac{\omega}{\omega_{11}}\right)^2\right]\left[1 - \left(\frac{\omega}{\omega_{22}}\right)^2\right] - \frac{k_2}{k_1}} \right\} \tag{52}$$

$$X_2 = \frac{\omega_{22}{}^2 F_1 / k_1}{\left[\omega_{22}{}^2 - \omega^2\right]} \left\{ \frac{1 - \left(\frac{\omega}{\omega_{22}}\right)^2}{\left[1 + \frac{k_2}{k_1} - \left(\frac{\omega}{\omega_{11}}\right)^2\right]\left[1 - \left(\frac{\omega}{\omega_{22}}\right)^2\right] - \frac{k_2}{k_1}} \right\} \tag{53}$$

$$X_2 = \frac{F_1 / k_1}{1 - \left(\frac{\omega}{\omega_{22}}\right)^2} \left\{ \frac{1 - \left(\frac{\omega}{\omega_{22}}\right)^2}{\left[1 + \frac{k_2}{k_1} - \left(\frac{\omega}{\omega_{11}}\right)^2\right]\left[1 - \left(\frac{\omega}{\omega_{22}}\right)^2\right] - \frac{k_2}{k_1}} \right\} \tag{54}$$

$$X_2 = \frac{F_1/k_1}{1-\left(\frac{\omega}{\omega_{22}}\right)^2}\left\{\frac{1-\left(\frac{\omega}{\omega_{22}}\right)^2}{\left[1+\frac{k_2}{k_1}-\left(\frac{\omega}{\omega_{11}}\right)^2\right]\left[1-\left(\frac{\omega}{\omega_{22}}\right)^2\right]-\frac{k_2}{k_1}}\right\} \tag{55}$$

$$X_2 = \left\{\frac{F_1/k_1}{\left[1+\frac{k_2}{k_1}-\left(\frac{\omega}{\omega_{11}}\right)^2\right]\left[1-\left(\frac{\omega}{\omega_{22}}\right)^2\right]-\frac{k_2}{k_1}}\right\} \tag{56}$$

Note that for $\omega = \omega_{22} = \sqrt{k_2/m_2}$,

$$X_1 = 0 \tag{57}$$

$$X_2 = -F_1/k_2 \tag{58}$$

The force acting on m_2 is: $-F_1$

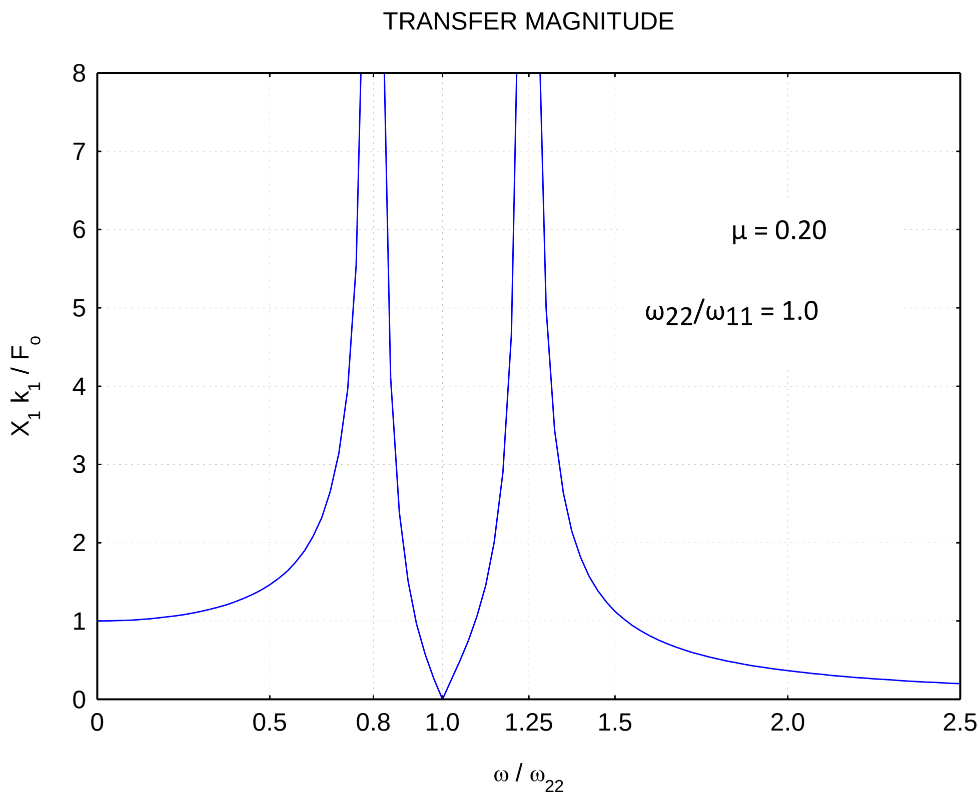

Figure 4.

The curve in Figure 4 is taken from equation (50).

Let

$$\mu = \frac{m_2}{m_1} \tag{59}$$

$$\frac{k_2}{k_1} = \mu \left(\frac{\omega_{22}}{\omega_{11}} \right)^2 \tag{60}$$

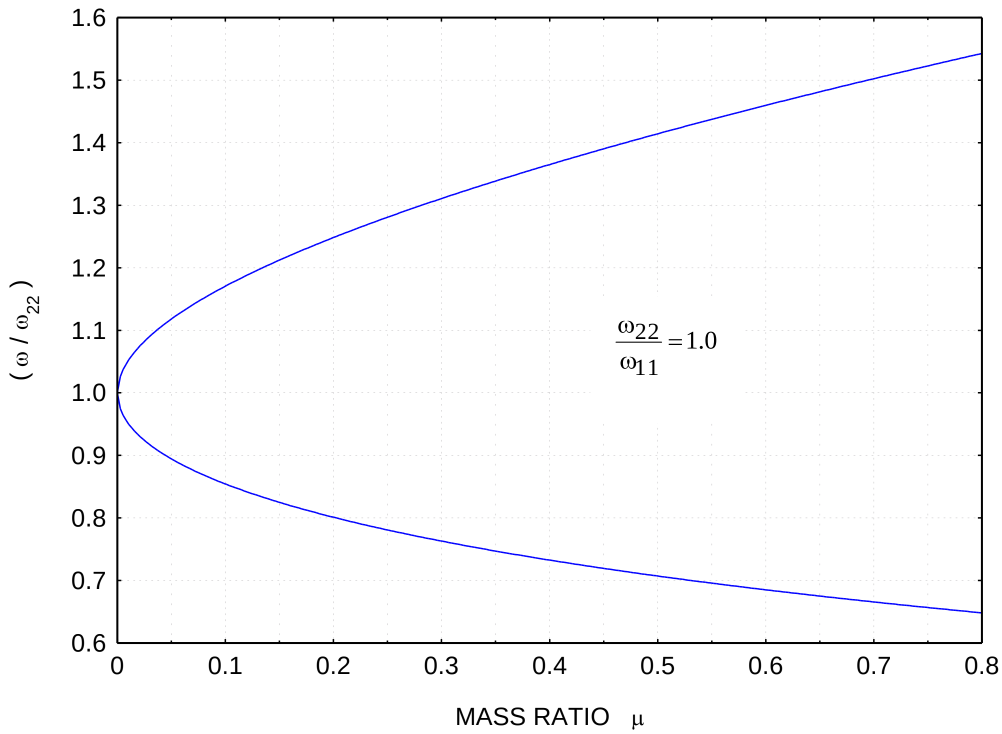

Figure 5.

The natural frequencies are shown as a function of mass ratio in Figure 5.

System with Damping

The derivation for a system with damping is given in Appendix A.

References

1. W. Thomson, Theory of Vibrations with Applications, Second Edition, Prentice-Hall, New Jersey, 1981.

2. R. Vierck, Vibration Analysis, 2nd Edition, Harper Collins, New York, 1979.

APPENDIX A

Two-degree-of-freedom System with Damping

Consider the system in Figure A-1.

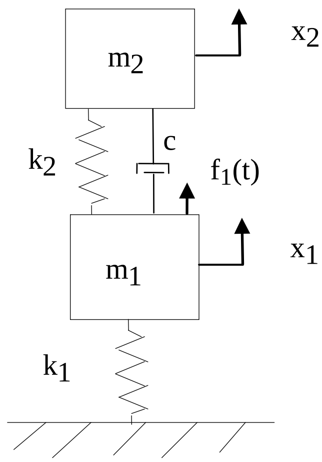

Figure A-1.

A free-body diagram of mass 1 is given in Figure A-2. A free-body diagram of mass 2 is given in Figure A-3.

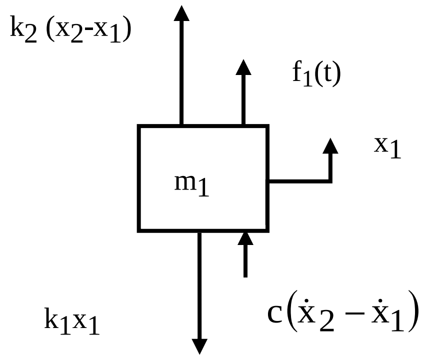

Figure A-2.

Determine the equation of motion for mass 1.

$$\sum F = m_1 \ddot{x}_1 \quad \text{(A-1)}$$

$$m_1\ddot{x}_1 = f_1(t) + c\left(\dot{x}_2 - \dot{x}_1\right) + k_2(x_2 - x_1) - k_1x_1 \quad \text{(A-2)}$$

$$m_1\ddot{x}_1 - c\left(\dot{x}_2 - \dot{x}_1\right) - k_2(x_2 - x_1) + k_1x_1 = f_1(t) \quad \text{(A-3)}$$

$$m_1\ddot{x}_1 + c\left(-\dot{x}_2 + \dot{x}_1\right) + k_2(-x_2 + x_1) + k_1x_1 = f_1(t) \quad \text{(A-4)}$$

$$m_1\ddot{x}_1 + c\dot{x}_1 - c\dot{x}_2 + (k_1 + k_2)x_1 - k_2x_2 = f_1(t) \quad \text{(A-5)}$$

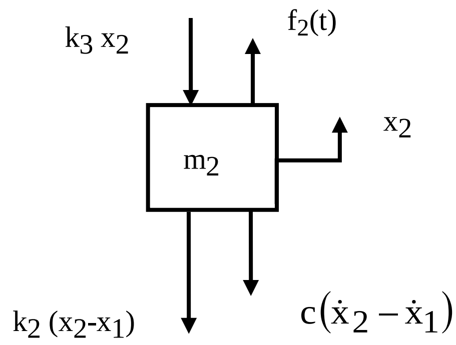

Figure A-3.

Derive the equation of motion for mass 2.

$$\sum F = m_2\,\ddot{x}_2 \tag{A-6}$$

$$m_2\ddot{x}_2 = -c(\dot{x}_2 - \dot{x}_1) - k_2(x_2 - x_1) \tag{A-7}$$

$$m_2\ddot{x}_2 + c(\dot{x}_2 - \dot{x}_1) + k_2(x_2 - x_1) = 0 \tag{A-8}$$

$$m_2\ddot{x}_2 + c\,\dot{x}_2 - c\,\dot{x}_1 + k_2 x_2 - k_2 x_1 = 0 \tag{A-9}$$

Assemble the equations in matrix form.

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} c & -c \\ -c & c \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} f_1(t) \\ 0 \end{bmatrix}$$

(A-10)

The homogenous, undamped form is

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix}+\begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{A-11}$$

Seek a solution of the form

$$\bar{x}=\bar{q}\exp(j\omega t) \tag{A-12}$$

The q vector is the generalized coordinate vector.

Note that

$$\dot{\bar{x}}=j\omega\bar{q}\exp(j\omega t) \tag{A-13}$$

$$\ddot{\bar{x}}=-\omega^2\,\bar{q}\exp(j\omega t) \tag{A-14}$$

The generalized eigenvalue problem is

$$\left\{\begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}-\omega^2\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\right\}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}\exp(j\omega t)=0 \tag{A-15}$$

$$\det\left\{\begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}-\omega^2\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\right\}=0 \tag{A-16}$$

$$\det\begin{bmatrix} k_1+k_2-\omega^2 m_1 & -k_2 \\ -k_2 & k_2-\omega^2 m_2 \end{bmatrix}=0 \tag{A-17}$$

$$\left(k_1+k_2-\omega^2 m_1\right)\left(k_2-\omega^2 m_2\right)-k_2{}^2=0 \tag{A-18}$$

$$k_1k_2 + k_2^2 - \omega^2 m_1 k_2 - \omega^2 m_2 k_1 - \omega^2 m_2 k_2 + \omega^4 m_1 m_2 = 0 \qquad \text{(A-19)}$$

$$\omega^4 m_1 m_2 - \omega^2 (m_1 k_2 + m_2 k_1 + m_2 k_2) + k_1 k_2 = 0 \qquad \text{(A-20)}$$

$$\omega^4 m_1 m_2 - \omega^2 [m_1 k_2 + m_2 (k_1 + k_2)] + k_1 k_2 = 0 \qquad \text{(A-21)}$$

The eigenvalues are the roots of the polynomial.

$$\omega_1^2 = \frac{-b - \sqrt{b^2 - 4ad}}{2a} \qquad \text{(A-22)}$$

$$\omega_2^2 = \frac{-b + \sqrt{b^2 - 4ad}}{2a} \qquad \text{(A-23)}$$

where

$$a = m_1 m_2 \qquad \text{(A-24)}$$

$$b = -[m_1 k_2 + m_2 (k_1 + k_2)] \qquad \text{(A-25)}$$

$$d = k_1 k_2 \qquad \text{(A-26)}$$

Frequency Response Function

Let

$$f_1(t) = F_1 \exp(j\omega t) \qquad \text{(A-27)}$$

$$x_1(t) = X_1 \exp(j\omega t) \qquad \text{(A-28)}$$

$$x_2(t) = X_2 \exp(j\omega t) \tag{A-29}$$

$$-\omega^2 m_1 X_1 \exp(j\omega t) + j\omega c X_1 \exp(j\omega t) + (k_1 + k_2) X_1 \exp(j\omega t)$$
$$- j\omega c X_2 \exp(j\omega t) - k_2 X_2 \exp(j\omega t) = F_1 \exp(j\omega t)$$

(A-30)

$$-\omega^2 m_1 X_1 + j\omega c X_1 + (k_1 + k_2) X_1 - j\omega c X_2 - k_2 X_2 = F_1 \tag{A-31}$$

$$\left[(k_1 + k_2) - \omega^2 m_1 + j\omega c\right] X_1 - \left[k_2 + j\omega c\right] X_2 = F_1 \tag{A-32}$$

Recall the equation for mass 2.

$$m_2 \ddot{x}_2 + c\dot{x}_2 - c\dot{x}_1 + k_2 x_2 - k_2 x_1 = 0 \tag{A-33}$$

$$-\omega^2 m_2 X_2 + jc\omega X_2 + k_2 X_2 - jc\omega X_2 - k_2 X_1 = 0 \tag{A-34}$$

$$\left[k_2 - \omega^2 m_2 + jc\omega\right] X_2 - \left[k_2 + jc\omega\right] X_1 = 0 \tag{A-35}$$

$$X_2 = \left[\frac{k_2 + jc\omega}{\left[k_2 - \omega^2 m_2 + jc\omega\right]}\right] X_1 \tag{A-36}$$

By substitution,

$$\left[(k_1+k_2)-\omega^2 m_1+jc\omega\right]X_1-\left[\frac{k_2\left[k_2+jc\omega\right]}{\left[k_2-\omega^2 m_2+jc\omega\right]}\right]X_1=F_1 \quad \text{(A-37)}$$

$$\left[(k_1+k_2)-\omega^2 m_1+jc\omega\right]\left[k_2-\omega^2 m_2+jc\omega\right]X_1-k_2\left[k_2+jc\omega\right]X_1$$
$$=F_1\left[k_2-\omega^2 m_2+jc\omega\right]$$

(A-38)

$$\left\{\left[(k_1+k_2)-\omega^2 m_1+jc\omega\right]\left[k_2-\omega^2 m_2+jc\omega\right]-k_2\left[k_2+jc\omega\right]\right\}X_1$$
$$=F_1\left[k_2-\omega^2 m_2+jc\omega\right]$$

(A-39)

$$X_1=\left[\frac{F_1\left[k_2-\omega^2 m_2+jc\omega\right]}{\left[(k_1+k_2)-\omega^2 m_1+jc\omega\right]\left[k_2-\omega^2 m_2+jc\omega\right]-k_2\left[k_2+jc\omega\right]}\right]$$

(A-40)

$$\frac{X_1}{F_1}=\left[\frac{\left[k_2-\omega^2 m_2+jc\omega\right]}{\left[(k_1+k_2)-\omega^2 m_1+jc\omega\right]\left[k_2-\omega^2 m_2+jc\omega\right]-k_2\left[k_2+jc\omega\right]}\right] \quad \text{(A-41)}$$

$$\frac{X_1}{F_1}=\frac{\left[k_2-\omega^2 m_2+jc\omega\right]}{k_1\left\{\left[1+\frac{k_2}{k_1}-\omega^2\frac{m_1}{k_1}+j\frac{c\omega}{k_1}\right]\left[k_2-\omega^2 m_2+jc\omega\right]-k_2\left[\frac{k_2}{k_1}+j\frac{c\omega}{k_1}\right]\right\}}$$

(A-42)

$$\frac{k_1 X_1}{F_1} = \frac{\left| k_2 - \omega^2 m_2 + jc\omega \right|}{\left[1 + \frac{k_2}{k_1} - \omega^2 \frac{m_1}{k_1} + j\frac{c\omega}{k_1}\right]\left[k_2 - \omega^2 m_2 + jc\omega\right] - k_2\left[\frac{k_2}{k_1} + j\frac{c\omega}{k_1}\right]} \tag{A-43}$$

$$\frac{k_1 X_1}{F_1} = \frac{\left| k_2 - \omega^2 m_2 + jc\omega \right|}{k_2\left[1 + \frac{k_2}{k_1} - \omega^2 \frac{m_1}{k_1} + j\frac{c\omega}{k_1}\right]\left[1 - \omega^2 \frac{m_2}{k_2} + j\frac{c\omega}{k_2}\right] - k_2\left[\frac{k_2}{k_1} + j\frac{c\omega}{k_1}\right]} \tag{A-44}$$

$$\frac{k_1 X_1}{F_1} = \frac{\left[1 - \omega^2 \frac{m_2}{k_2} + j\frac{c\omega}{k_2}\right]}{\left[1 + \frac{k_2}{k_1} - \omega^2 \frac{m_1}{k_1} + j\frac{c\omega}{k_1}\right]\left[1 - \omega^2 \frac{m_2}{k_2} + j\frac{c\omega}{k_2}\right] - \left[\frac{k_2}{k_1} + j\frac{c\omega}{k_1}\right]} \tag{A-45}$$

Let

$$\omega_{11}{}^2 = \frac{k_1}{m_1} \tag{A-46}$$

$$\omega_{22}{}^2 = \frac{k_2}{m_2} \tag{A-47}$$

$$\frac{k_1 X_1}{F_1} = \frac{\left[1 - \frac{\omega^2}{\omega_{22}{}^2} + j\frac{c\omega}{k_2}\right]}{\left[1 + \frac{k_2}{k_1} - \frac{\omega^2}{\omega_{11}{}^2} + j\frac{c\omega}{k_1}\right]\left[1 - \frac{\omega^2}{\omega_{22}{}^2} + j\frac{c\omega}{k_2}\right] - \left[\frac{k_2}{k_1} + j\frac{c\omega}{k_1}\right]} \tag{A-48}$$

Let

$$\mu = \frac{m_2}{m_1} \tag{A-49}$$

$$\frac{k_2}{k_1} = \mu\left(\frac{\omega_{22}}{\omega_{11}}\right)^2 \tag{A-50}$$

$$\frac{k_1 X_1}{F_1} = \frac{\left[1 - \frac{\omega^2}{\omega_{22}^{\;2}} + j\frac{c\omega}{k_2}\right]}{\left[1 + \mu\left(\frac{\omega_{22}}{\omega_{11}}\right)^2 - \frac{\omega^2}{\omega_{11}^{\;2}} + j\frac{c\omega}{k_1}\right]\left[1 - \frac{\omega^2}{\omega_{22}^{\;2}} + j\frac{c\omega}{k_2}\right] - \left[\mu\left(\frac{\omega_{22}}{\omega_{11}}\right)^2 + j\frac{c\omega}{k_1}\right]}$$

(A-51)

$$\frac{k_1 X_1}{F_1} =$$

$$\frac{\left[1 - \frac{\omega^2}{\omega_{22}^{\;2}} + j\left(\frac{c\omega}{\omega_{22}^{\;2} m_2}\right)\right]}{\left[1 + \mu\left(\frac{\omega_{22}}{\omega_{11}}\right)^2 - \frac{\omega^2}{\omega_{11}^{\;2}} + j\left(\frac{\mu c\omega}{\omega_{11}^{\;2} m_2}\right)\right]\left[1 - \frac{\omega^2}{\omega_{22}^{\;2}} + j\left(\frac{c\omega}{\omega_{22}^{\;2} m_2}\right)\right] - \left[\mu\left(\frac{\omega_{22}}{\omega_{11}}\right)^2 + j\left(\frac{\mu c\omega}{\omega_{11}^{\;2} m_2}\right)\right]}$$

(A-52)

Let

$$\frac{c}{m_2} = 2\xi\omega_{22} \tag{A-53}$$

$$\frac{k_1 X_1}{F_1} =$$

$$\frac{\left[1-\frac{\omega^2}{\omega_{22}{}^2}+j2\xi\left(\frac{\omega}{\omega_{22}}\right)\right]}{\left[1+\mu\left(\frac{\omega_{22}}{\omega_{11}}\right)^2-\frac{\omega^2}{\omega_{11}{}^2}+j2\xi\mu\left(\frac{\omega_{22}}{\omega_{11}}\right)\left(\frac{\omega}{\omega_{11}}\right)\right]\left[1-\frac{\omega^2}{\omega_{22}{}^2}+j2\xi\left(\frac{\omega}{\omega_{22}}\right)\right]-\left[\mu\left(\frac{\omega_{22}}{\omega_{11}}\right)^2+j2\xi\mu\left(\frac{\omega_{22}}{\omega_{11}}\right)\left(\frac{\omega}{\omega_{11}}\right)\right]}$$

(A-54)

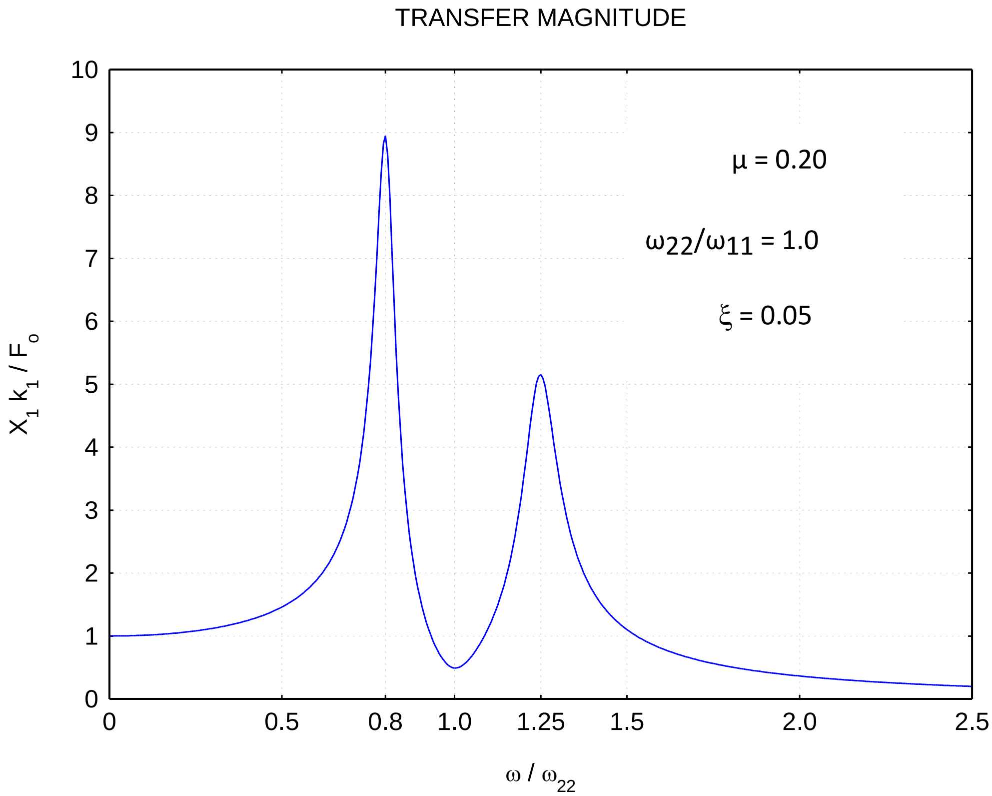

Figure A-4.

Equation (A-54) is plotted in Figure A-4 for a particular case.