DERIVATION OF THE VOLUME OF A CONE AND OF A CONICAL SHELL   Revision A

By Tom Irvine
Email: tomirvine@aol.com

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Volume of a Cone

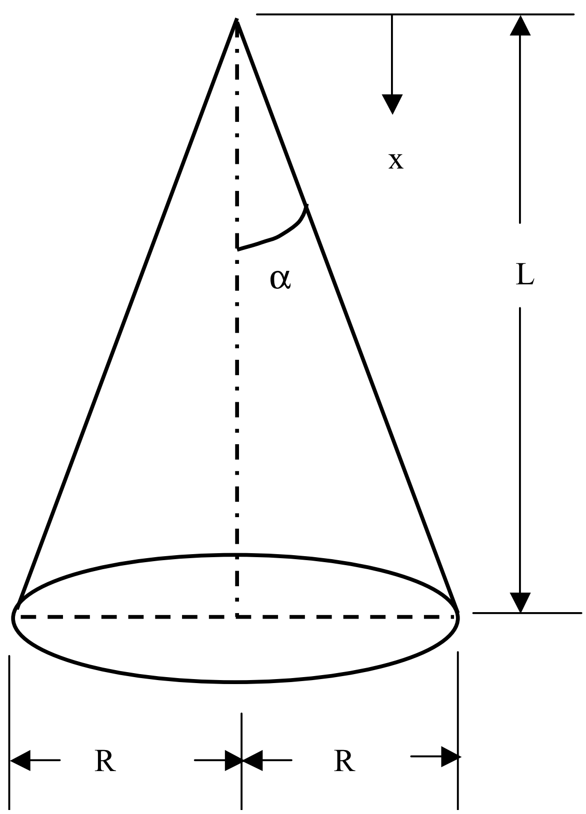

Figure 1.

The volume V of the solid circular right cone is

$$V = \int_0^{2\pi} \int_0^{L} \int_0^{xR/L} r\,dr\,dx\,d\theta \tag{1.1}$$

$$V = \frac{1}{2}\int_0^{2\pi}\int_0^{L} r^2 \Big|_0^{xR/L} dxd\theta \tag{1.2}$$

$$V = \frac{1}{2}\int_0^{2\pi}\int_0^{L} \left[x\left(\frac{R}{L}\right)\right]^2 dxd\theta \tag{1.3}$$

$$V = \frac{1}{2}\left(\frac{R}{L}\right)^2 \int_0^{2\pi}\int_0^{L} x^2\, dxd\theta \tag{1.4}$$

$$V = \frac{1}{2}\left(\frac{R}{L}\right)^2 \int_0^{2\pi} \frac{1}{3}x^3 \Big|_0^{L} d\theta \tag{1.5}$$

$$V = \frac{1}{6}\left(\frac{R}{L}\right)^2 L^3 \int_0^{2\pi} d\theta \tag{1.6}$$

$$V = \frac{1}{6}R^2 L \int_0^{2\pi} d\theta \tag{1.7}$$

$$V = \frac{1}{6}R^2 L(2\pi) \tag{1.8}$$

$$V = \frac{1}{3}\pi R^2 L \tag{1.9}$$

Volume of a Conical Shell

Let h equal the thickness.

Let $\hat{h}$ equal the apparent thickness component parallel to the base.

$$V \approx \int_0^{2\pi} \int_0^L \int_{(xR/L)-\hat{h}}^{xR/L} r\,dr\,dx\,d\theta \tag{2.1}$$

$$V \approx \frac{1}{2} \int_0^{2\pi} \int_0^L r^2 \Big|_{(xR/L)-\hat{h}}^{xR/L} dx\,d\theta \tag{2.2}$$

$$V \approx \frac{1}{2} \int_0^{2\pi} \int_0^L \left\{ \left[x\left(\frac{R}{L}\right) \right]^2 - \left[x\left(\frac{R}{L}\right) - \hat{h} \right]^2 \right\} dx\,d\theta \tag{2.3}$$

$$V \approx \frac{1}{2} \int_0^{2\pi} \int_0^L \left\{ \left[x\left(\frac{R}{L}\right) \right]^2 - \left[\left[x\left(\frac{R}{L}\right) \right]^2 - 2\hat{h}x\left(\frac{R}{L}\right) + \hat{h}^2 \right] \right\} dx\,d\theta \tag{2.4}$$

$$V \approx \frac{1}{2} \int_0^{2\pi} \int_0^L \left\{ \left[x\left(\frac{R}{L}\right) \right]^2 - \left[x\left(\frac{R}{L}\right) \right]^2 + 2\hat{h}x\left(\frac{R}{L}\right) - \hat{h}^2 \right\} dx\,d\theta \tag{2.5}$$

$$V \approx \frac{1}{2} \int_0^{2\pi} \int_0^L \left[2\hat{h}x\left(\frac{R}{L}\right) - \hat{h}^2 \right] dx\,d\theta \tag{2.6}$$

$$V \approx \frac{\hat{h}}{2} \int_0^{2\pi} \left[x^2\left(\frac{R}{L}\right) - \hat{h}x \right]_0^L d\theta \tag{2.7}$$

$$V \approx \frac{\hat{h}}{2} \int_0^{2\pi} \left[L^2 \left(\frac{R}{L} \right) - \hat{h} L \right] d\theta \tag{2.8}$$

$$V \approx \frac{\hat{h} L}{2} \int_0^{2\pi} \left[R - \hat{h} \right] d\theta \tag{2.9}$$

$$V \approx \frac{\hat{h} L}{2} 2\pi \left[R - \hat{h} \right] \tag{2.10}$$

$$V \approx \pi \hat{h} L \left[R - \hat{h} \right] \tag{2.11}$$

For a thin shell $R >> \hat{h}$,

$$V \approx \pi \hat{h} L R \tag{2.12}$$

The apparent thickness is related to the actual thickness by

$$\hat{h} = \frac{h}{\cos \alpha} \tag{2.13}$$

Substitute equation (2.13) into (2.13).

$$V \approx \pi \frac{h L R}{\cos \alpha} \tag{2.14}$$

$$V \approx \pi \frac{h L R \sqrt{L^2 + R^2}}{L} \tag{2.15}$$

$$V \approx \pi h R \sqrt{L^2 + R^2} \tag{2.16}$$

Volume of a Conical Shell, Alternate Coordinate System

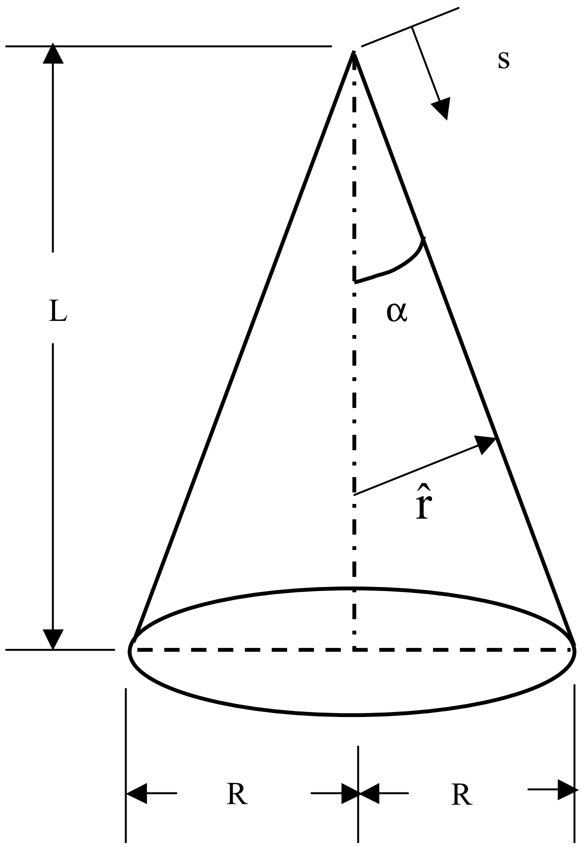

Figure 2.

$$V \approx \int_0^{2\pi} \int_0^{L \sec \alpha} \int_{s \tan \alpha - \hat{h}}^{s \tan \alpha} \hat{r} d\hat{r} \, ds d\theta \qquad (3.1)$$

$$V \approx \frac{1}{2} \int_0^{2\pi} \int_0^{L \sec \alpha} \hat{r}^2 \Big|_{s \tan \alpha - \hat{h}}^{s \tan \alpha} ds d\theta \qquad (3.2)$$

$$V \approx \frac{1}{2}\int_0^{2\pi}\int_0^{L\sec\alpha}\left\{[s\tan\alpha]^2 - \left[s\tan\alpha - \hat{h}\right]^2\right\}ds\,d\theta \tag{3.3}$$

$$V \approx \frac{1}{2}\int_0^{2\pi}\int_0^{L\sec\alpha}\left\{[s\tan\alpha]^2 - \left[[s\tan\alpha]^2 - 2\hat{h}s\tan\alpha + \hat{h}^2\right]\right\}ds\,d\theta \tag{3.4}$$

$$V \approx \frac{1}{2}\int_0^{2\pi}\int_0^{L\sec\alpha}\left\{[s\tan\alpha]^2 - [s\tan\alpha]^2 + 2\hat{h}s\tan\alpha - \hat{h}^2\right\}ds\,d\theta \tag{3.5}$$

$$V \approx \frac{1}{2}\int_0^{2\pi}\int_0^{L\sec\alpha}\left\{2\hat{h}s\tan\alpha - \hat{h}^2\right\}ds\,d\theta \tag{3.6}$$

$$V \approx \frac{1}{2}\int_0^{2\pi}\left[\hat{h}s^2\tan\alpha - \hat{h}^2 s\right]\Big|_0^{L\sec\alpha} d\theta \tag{3.7}$$

$$V \approx \frac{1}{2}\int_0^{2\pi}\left[\hat{h}(L\sec\alpha)^2\tan\alpha - \hat{h}^2(L\sec\alpha)\right]d\theta \tag{3.8}$$

$$V \approx \frac{1}{2}\hat{h}L\sec\alpha\int_0^{2\pi}\left[L\sec\alpha\tan\alpha - \hat{h}\right]d\theta \tag{3.9}$$

$$V \approx \frac{1}{2}\hat{h}L\sec\alpha\int_0^{2\pi}\left[L\sin\alpha - \hat{h}\right]d\theta \tag{3.10}$$

$$V \approx \pi\hat{h}L\sec\alpha\left[L\sin\alpha - \hat{h}\right] \tag{3.11}$$

For a thin shell $L \sin\alpha >> \hat{h}$,

$$V \approx \pi \hat{h} L^2 \sec\alpha \sin\alpha \tag{3.12}$$

$$\hat{h} = \frac{h}{\cos\alpha} \tag{3.13}$$

$$V \approx \pi h L^2 \left(\frac{\sec\alpha \sin\alpha}{\cos\alpha} \right) \tag{3.14}$$

$$V \approx \pi h L^2 \sec^2 \alpha \sin\alpha \tag{3.15}$$

$$V \approx \pi h L^2 \left(\frac{L^2 + R^2}{L^2} \right) \left(\frac{R}{\sqrt{L^2 + R^2}} \right) \tag{3.16}$$

$$V \approx \pi h R \sqrt{L^2 + R^2} \tag{3.17}$$