SHOCK TIME HISTORY RECONSTRUCTION VIA WAVELETS Revision B

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Introduction

The purpose of this paper is to give a method for synthesizing a time history using a wavelet series to represent a measured shock time history. The synthesized time history could then be applied as a base input on a shaker table to a test item.

Note that the wavelet series has zero net displacement and zero net velocity. This is a necessary characteristic for shaker shock tests. Furthermore, this condition is satisfied by each individual wavelet, as well as by the complete series.

This wavelet reconstruction method is alternative to traditional shock response spectrum (SRS) methods, in terms of test specification and fulfillment.

The wavelet approach may also be used as an extension of the SRS method, satisfying criteria both in the time and natural frequency domains.

Background

Most shock specifications in the aerospace industry are given in terms of a shock response spectrum. In some cases, specifications are also given in terms of classical base inputs. A third format is drop shock onto a hard surface from a prescribed height.

The SRS models the responses of individual single-degree-of-freedom systems to a common base input. The natural frequency of each system is an independent variable. The damping value is usually fixed at 5%, or equivalently at Q=10. The SRS calculation retains the peak response of each system as a function of natural frequency. The resulting SRS is plotted in terms of peak acceleration (G) versus natural frequency (Hz).

A given time history has a unique SRS. On the other hand, a given SRS may be satisfied by a variety of base inputs within prescribed tolerance bands.

Now consider that an avionics component mounted on some vehicle must withstand a complex oscillating pulse which has been measured during a field test, for example. The data might also come from a flight in the case of a missile or aircraft.

The avionics component must be tested in a lab to withstand this base input time history. The measured time history may or may not be reproducible in a test lab, however.

The measured time history can be converted into an SRS specification. The SRS method provides an indirect method for satisfying the specification by allowing for the substitution of a base input time history which is different than the one measured in the field test. The important point is that the test lab time history must have an SRS that matches the SRS of the field data within prescribed tolerance bands.

There is some concern in the aerospace industry, however, regarding the limitations of the SRS method. In particular, a given avionics component most likely responds as a multi-degree-of-freedom system. Cumulative fatigue and non-linear responses are additional concerns.

These problems can be largely solved by reproducing an actual measured time history in the lab, if possible. This reconstruction approach is discussed briefly in Reference 1. An excerpt from this reference is given in Appendix A.

Again, this reconstruction can be achieved via wavelets.[1]

Wavelet Method

For simplicity, assume that the measured time history is within the shaker's limits in terms of frequency and acceleration. Furthermore, assume that the control computer can accept a time history input in ASCII text format.

Ideally, the exact measured time history could then be input directly into the control computer. A measured time history, however, almost always has a non-zero net displacement which likely exceeds the shaker's limit. The resulting displacement may be real but is usually spurious.

The wavelet method overcomes this obstacle. Furthermore, the wavelet method yields a mathematically closed-form approximation of the measured signal.

[1] An alternative reconstruction could be made using damped sinusoids, but a compensation pulse would be required to ensure zero net displacement and zero net velocity.

The equation for an individual wavelet is

$$W_m(t) = \begin{cases} 0, \quad \text{for } t < t_{dm} \\ A_m \sin\left[\dfrac{2\pi f_m}{N_m}\left(t - t_{dm}\right)\right]\sin\left[2\pi f_m\left(t - t_{dm}\right)\right], \quad \text{for } t_{dm} \le t \le \left[t_{dm} + \dfrac{N_m}{2f_m}\right] \\ 0, \quad \text{for } t > \left[t_{dm} + \dfrac{N_m}{2f_m}\right] \end{cases} \tag{1}$$

where

$W_m(t)$	=	acceleration of wavelet m at time t
A_m	=	wavelet acceleration amplitude
f_m	=	wavelet frequency
N_m	=	number of half-sines
t_{dm}	=	wavelet time delay

Note that N_m must be an odd integer greater than or equal to 3.

The wavelet formula is well established in the vibration test industry, as shown in Reference 2 for example.

The corresponding velocity and displacement are derived in Appendices E and F, respectively. These appendices also give proof that each metric has a net value of zero.

The initial velocity and initial displacement are each zero for each wavelet.

The total acceleration at time t for a set of n wavelets is

$$\ddot{x}(t) = \sum_{m=1}^{n} W_m(t) \tag{2}$$

The coefficients required to match a given time history can be determined via brute force trial-and-error using random number generation. The approach is to select the wavelet that yields the lowest error when subtracted from the measured data. Over one hundred thousand iterations may be used for each wavelet.

The optimized wavelet is then subtracted from the measured signal for the next run. This process is then repeated for each additional wavelet.

First Example

A sample time history from a space shuttle solid rocket booster is shown in Figure 1a. The data was collected as the booster impacted the ocean. An avionics component mounted on the booster must withstand the water impact event because it must be re-used in future flights.

A series of sixty wavelets was synthesized to model the measured time history. The resulting parameters are shown in Appendix B. A further explanation of wavelets is also given in this appendix.

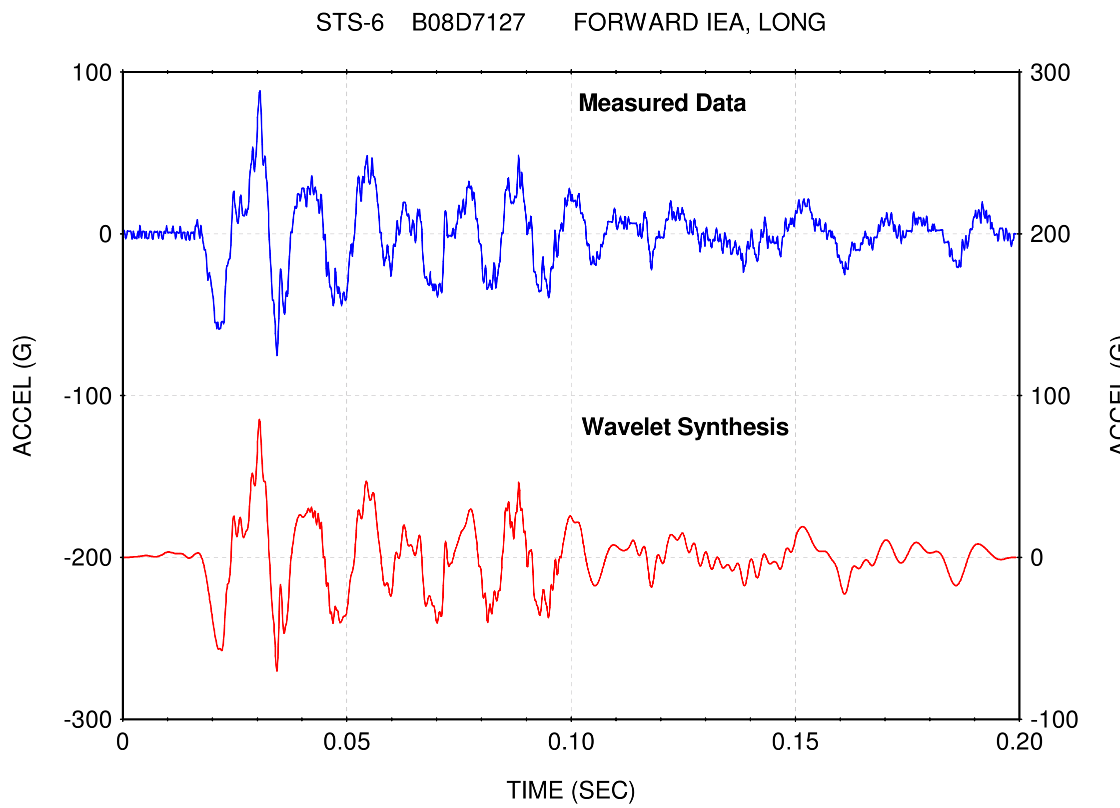

Figure 1a.

The synthesized time history is also shown in Figure 1a. The agreement is very good. The measured time history has some high frequency noise that was not modeled, however.

(Note that Figure 1a and the other graphs in this report are color plots.)

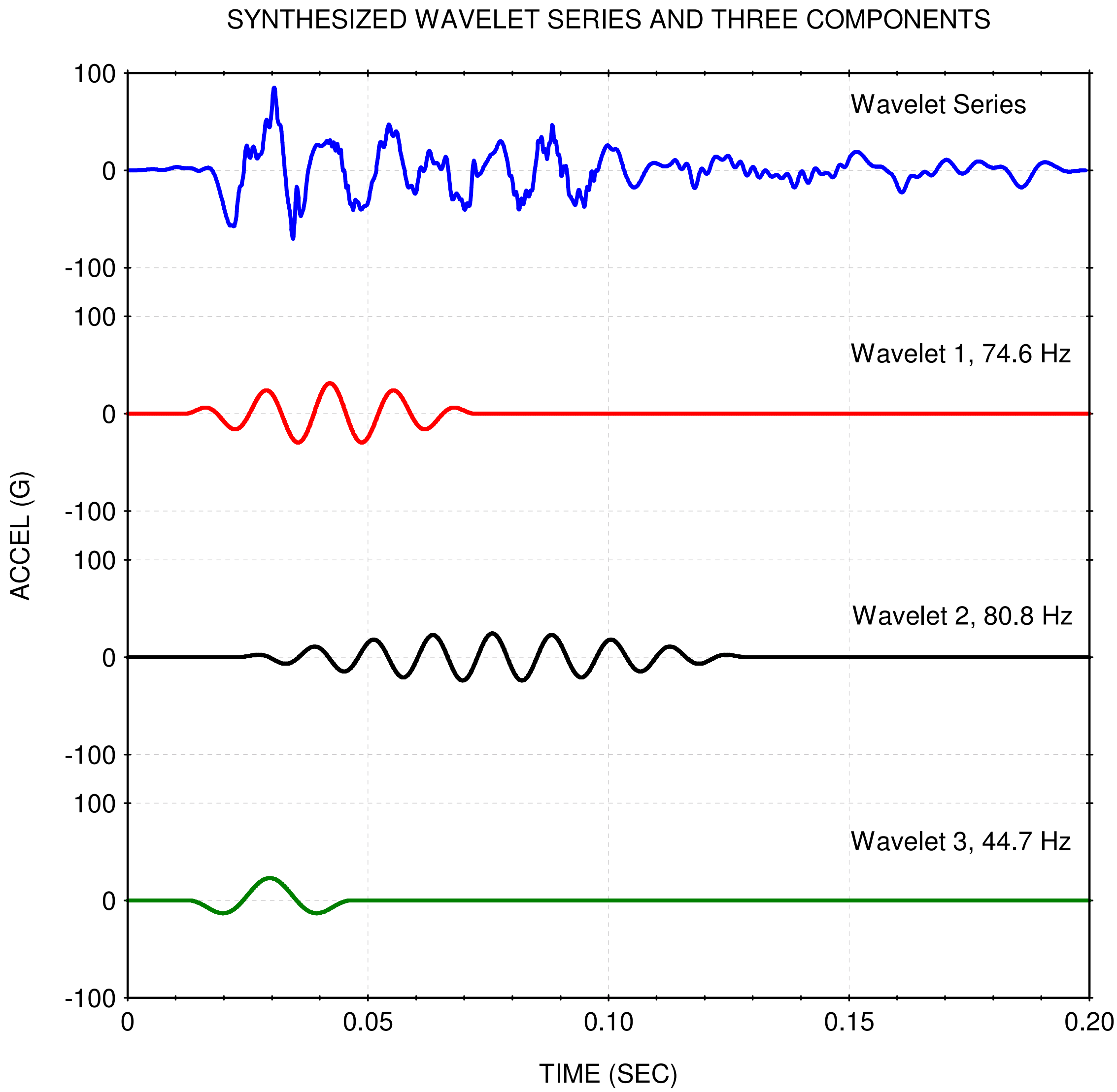

Figure 1b.

The synthesized time history in Figure 1a could be used as a basis for deriving a maximum expected environment (MEE). An appropriate statistical uncertainty margin should be added as a step in this process.

The synthesized waveform is shown in Figure 1b along with three of its components. The individual wavelet frequencies could be useful for identifying modal frequencies, complementing other tools such as the Fourier transform.

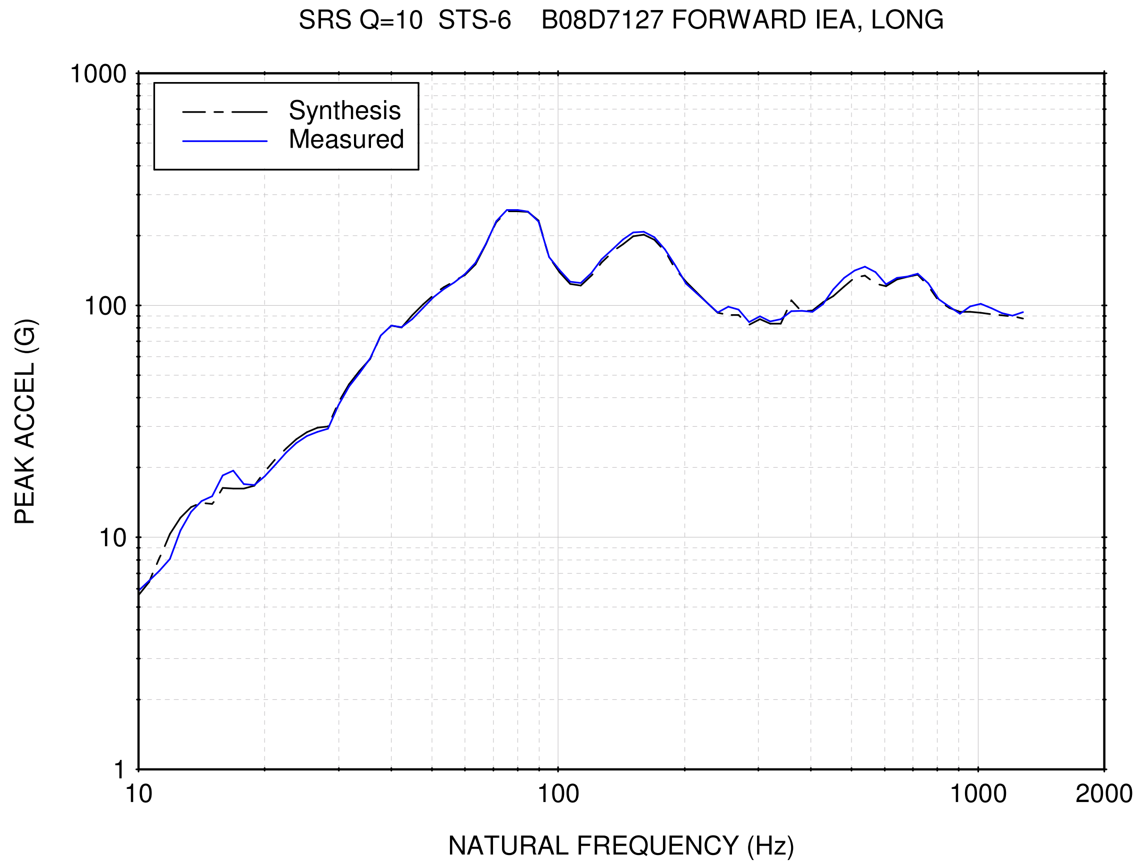

Figure 2.

The shock response spectra comparison is likewise very good.

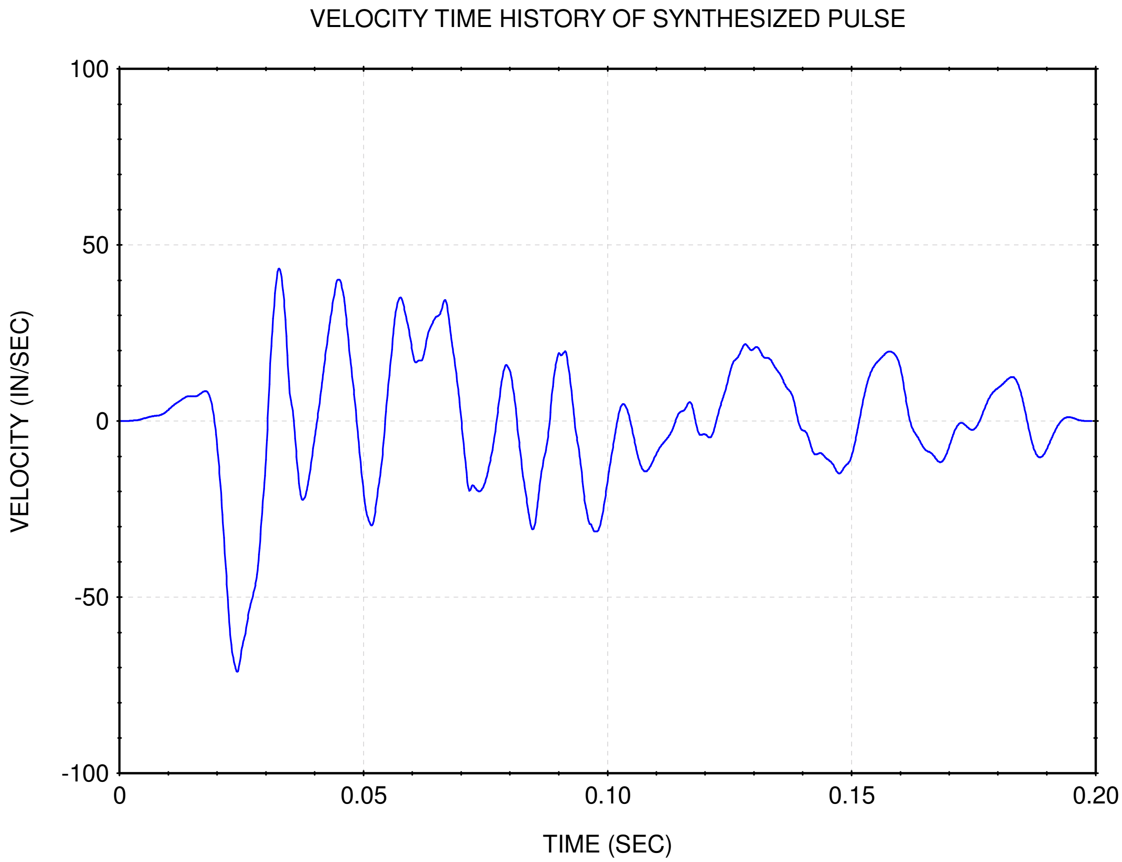

Figure 3.

The velocity time history integrated from the acceleration time history is shown in Figure 3. The net velocity is zero.

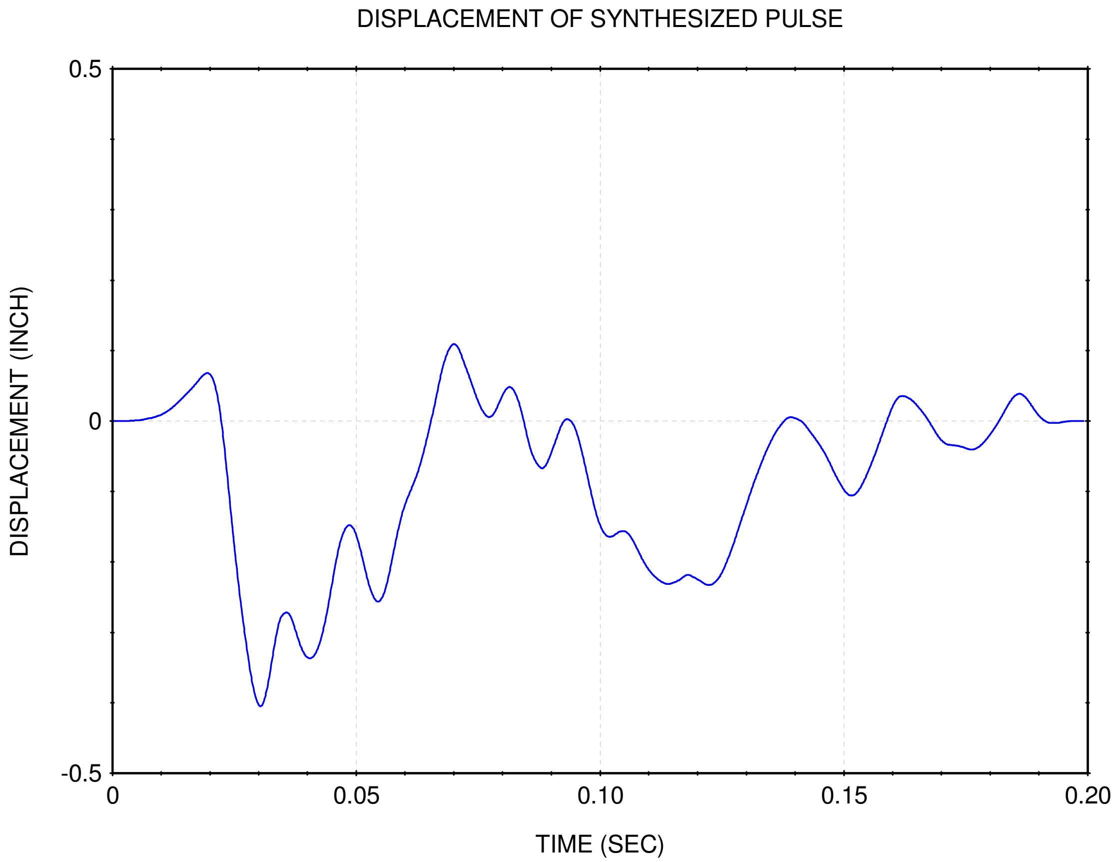

Figure 4.

The displacement time history double-integrated from the acceleration time history is shown in Figure 4. The net displacement is zero.

Second Example

Reconstructing a single time history via wavelets is relatively straightforward.

Now consider a complex case where up to three accelerometers were mounted adjacent to a large avionics component on the rocket booster. Again, the purpose was to record the shock for the water impact event.

The accelerometers were mounted in different locations but each in the longitudinal axis. The goal was to account for spatial variation. Furthermore, data was measured on each of two flights to account for flight-to-flight variation. This is important since the wind conditions, sea state, and other parameters may vary significantly from one flight to the next.

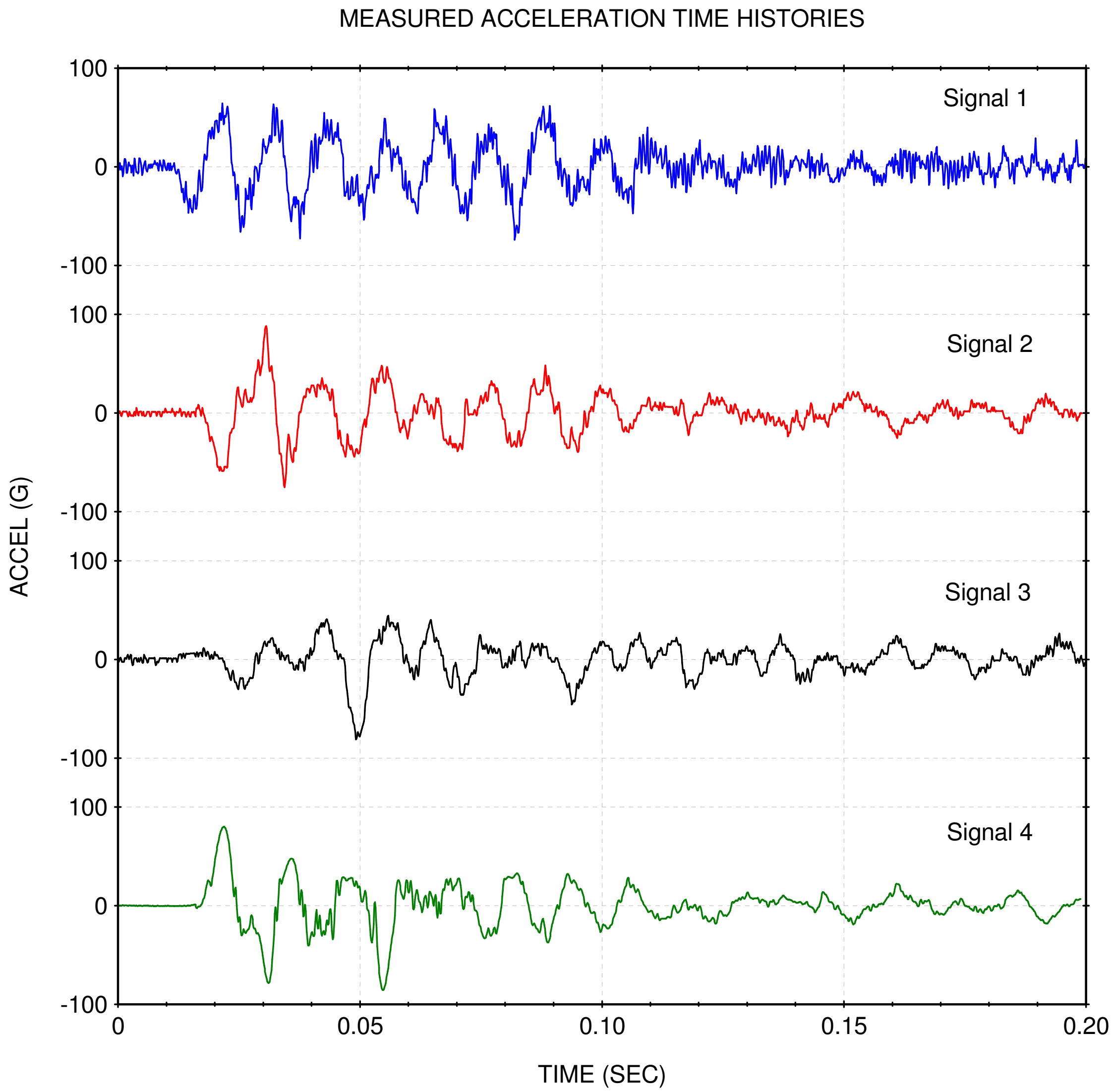

Figure 5.

Table 1. Forward IEA, Longitudinal Accelerometers		
Signal	Mission	S/N
1	STS-5	B08D8127
2	STS-6	B08D7127
3	STS-6	B08D8127
4	STS-6	B08D7110

The four measured time histories are shown in Figure 5. Note that signal 2 is the same as that shown in Figure 1. The raw data corresponding to the fourth signal appeared to be clipped. A cubic spline method was used to estimate the true signal.

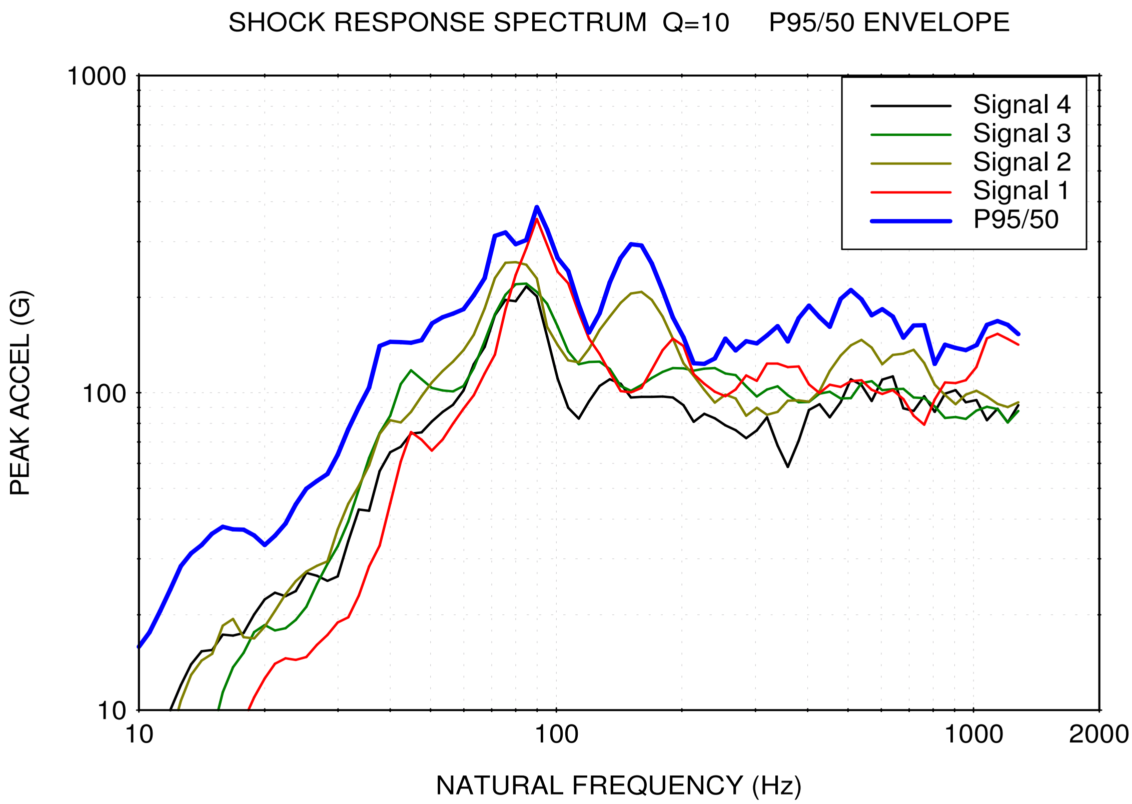

Figure 6.

The four shock response spectra are shown in Figure 6. A P95/50 envelope is also shown. The P95/50 method is taken from References 3 and 4. A brief summary is given in Appendix C.

The P95/50 method is one of several possible enveloping techniques for establishing an MEE level.

The wavelet reconstruction method may be used with other envelope types.

The next step is to derive a time history pulse which satisfies two goals:

1. The SRS of the synthesized pulse must match the P95/50 SRS within $\pm$3 dB tolerance bands.
2. The synthesized pulse must "resemble" the composite of the measured time histories in Figure 5.

The approach is:

1. Add the measured time histories to form a single composite pulse, as discussed in Appendix D.
2. Synthesize a wavelet time history to match the composite signal.
3. Calculate the SRS of the synthesized wavelet series.
4. Compare the wavelet SRS to the P95/50 SRS.
5. Scale the wavelet components appropriately so that the two SRS curves agree within tolerance bands.
6. Verify that the re-scaled synthesized time history "resembles" each of the measured signals.

Note that steps 3 through 5 are repeated over hundreds of iterations.

The resulting unscaled composite pulse for step 1 is shown in Figure 7.

The scaled synthesized wavelet acceleration pulse is given in Figure 8. The corresponding velocity and displacement time histories are given in Figures 9 and 10, respectively. The SRS comparison is given in Figure 11.

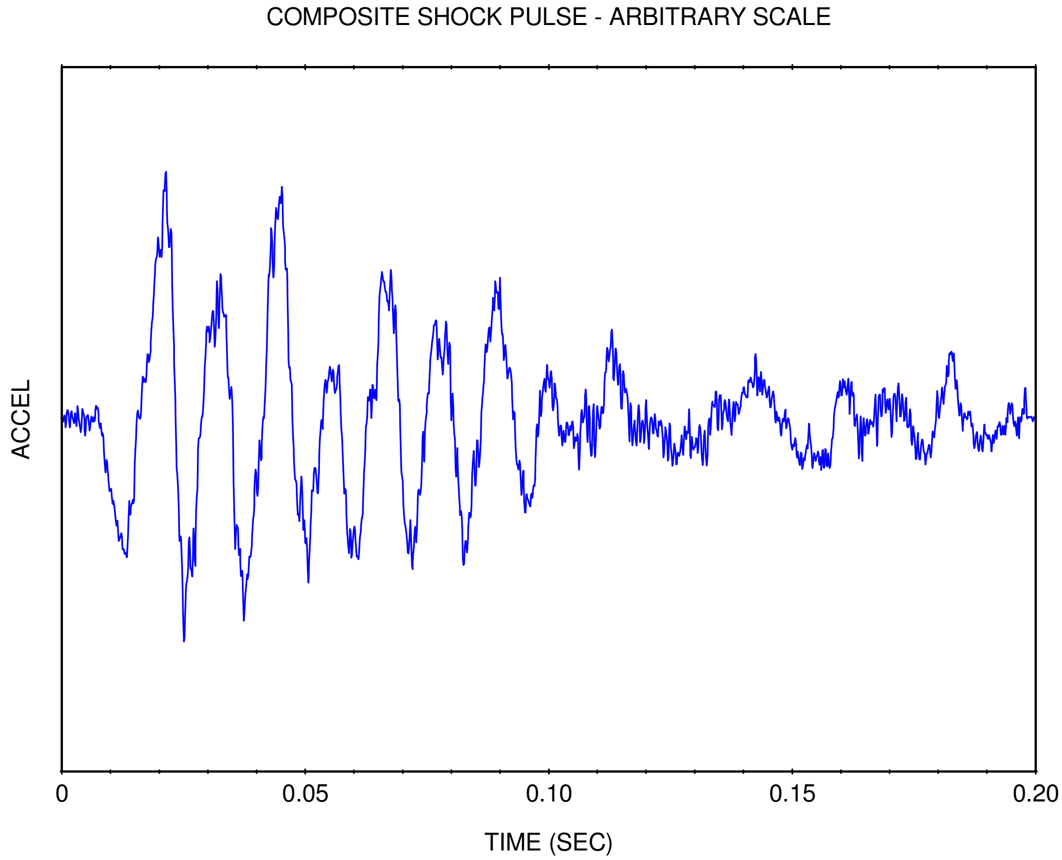

Figure 7.

This is the composite of the four measured signals shown in Figure 5. The scale is arbitrary. The composite consists of 120 individual wavelets.

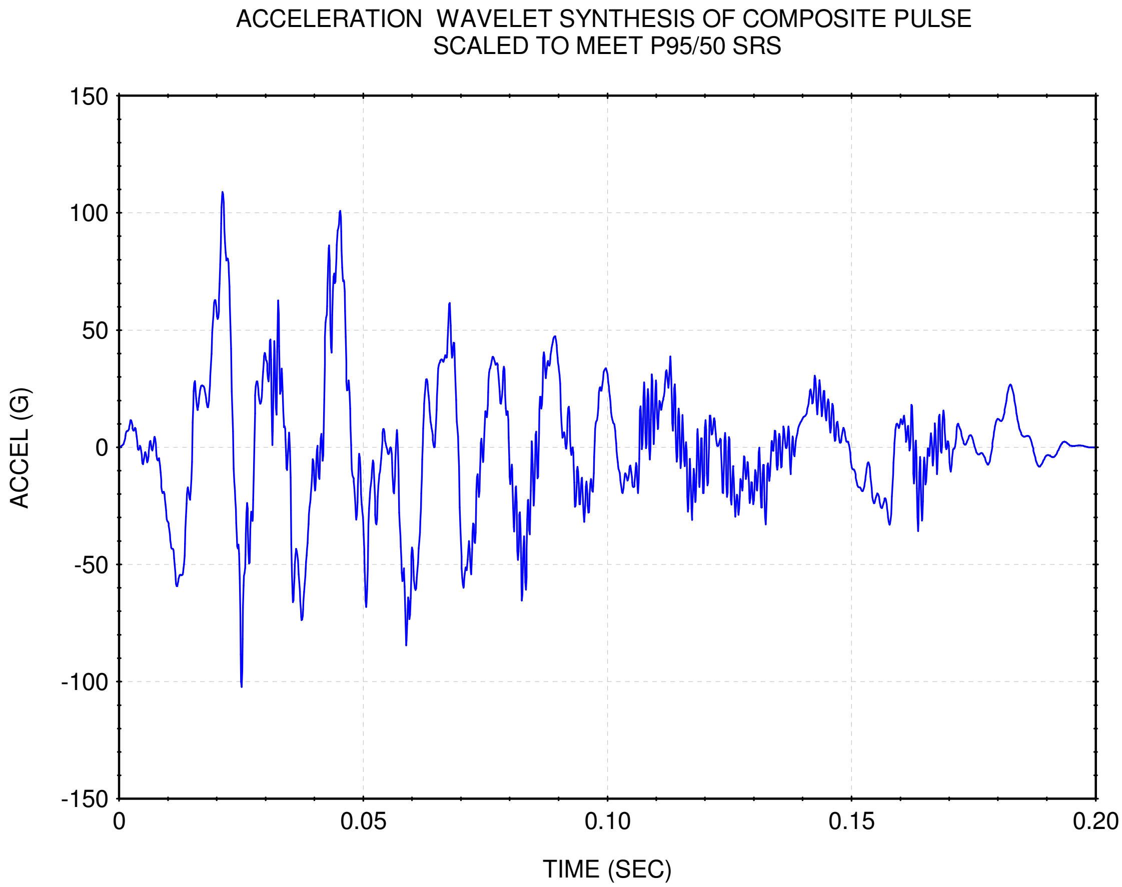

Figure 8.

The scaled wavelet pulse in Figure 8 qualitatively resembles the unscaled pulse in Figure 7.

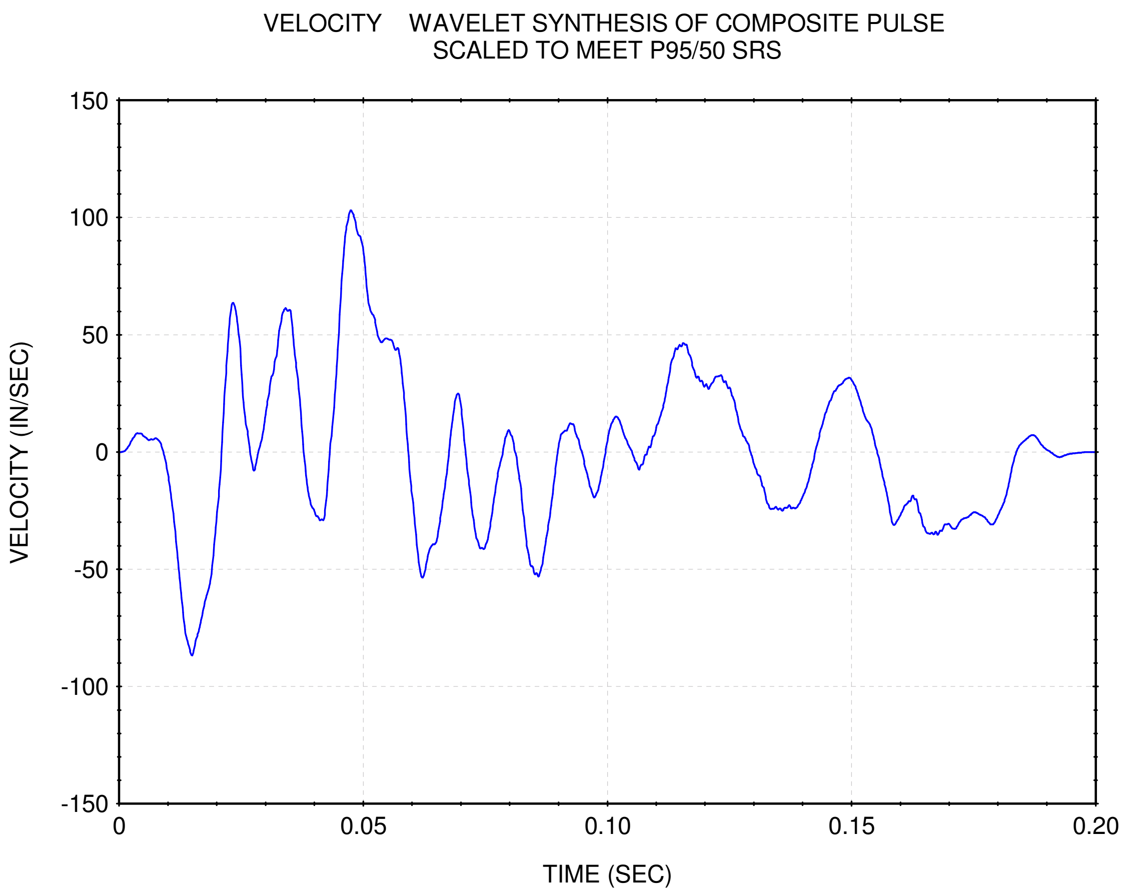

Figure 9.

The velocity pulse is integrated from the acceleration pulse. The net velocity is zero.

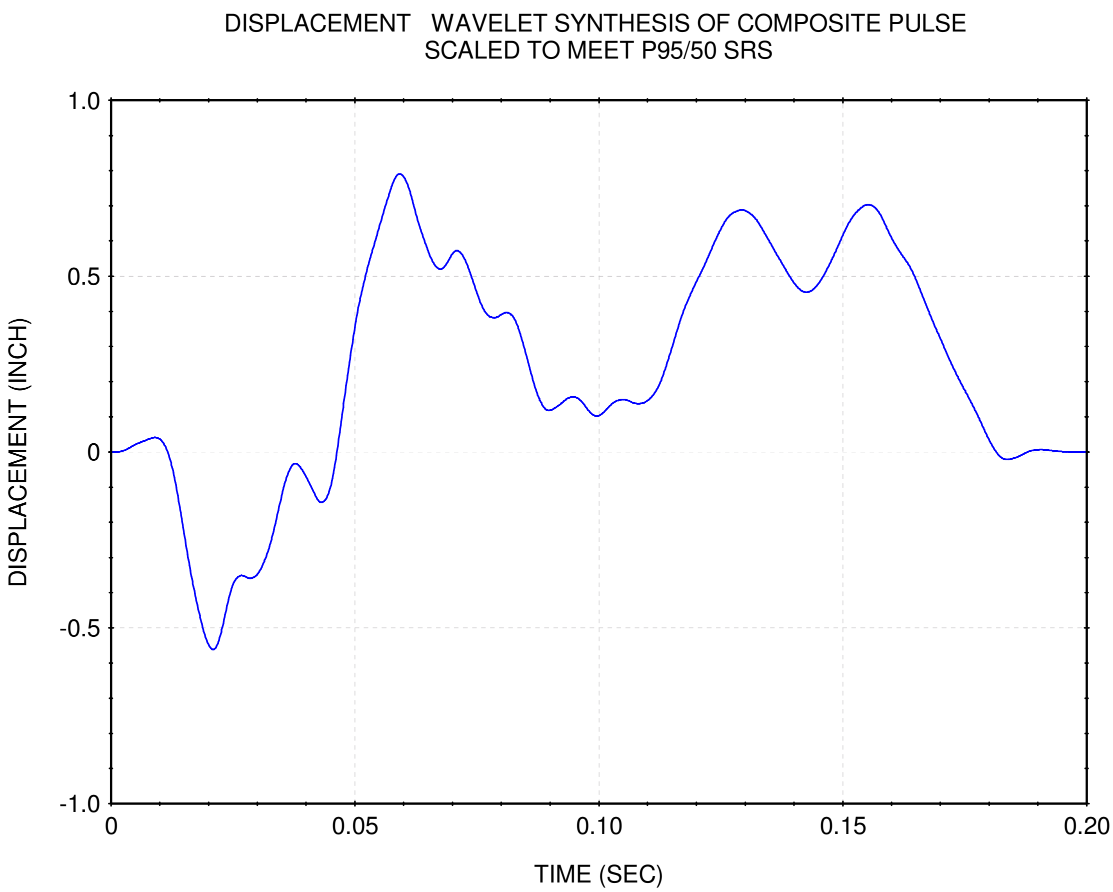

Figure 10.

The net displacement is zero.

The peak displacement may be too high for certain shaker tables. It is based on an SRS specification that has a starting frequency at 10 Hz. The peak displacement could be reduced if the specification were to begin at 20 Hz, for example. Furthermore, optimization could be performed to reduce the peak displacement while still meeting the goals of "acceleration time history resemblance" and SRS fulfillment.

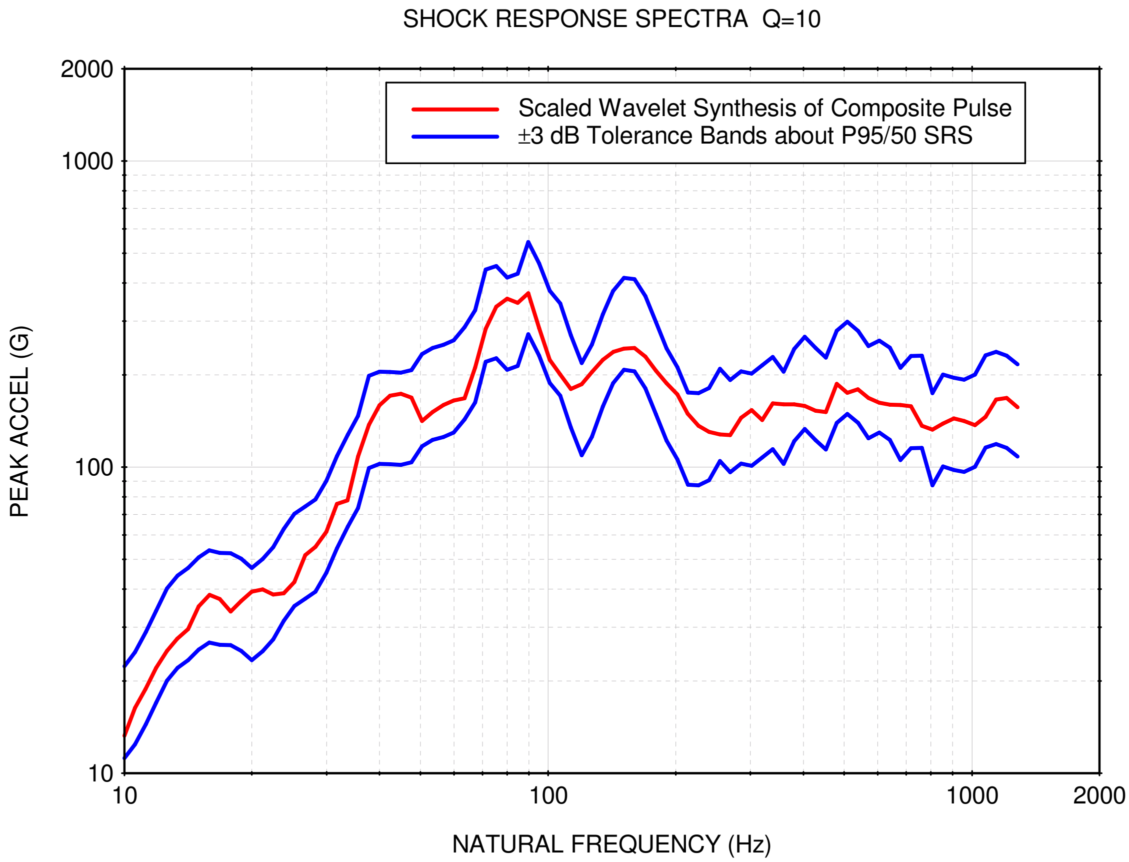

Figure 11.

The agreement is very good.

Conclusion

This paper presented a method for synthesizing a time history to represent a measured time history using wavelets.

In addition, the method was extended for the case of multiple measured time histories.

A summary of the software programs used in the examples is given in Appendix G.

The wavelet time history may be applied to a test item via a shaker table with suitable frequency and amplitude limits.

Furthermore, the synthesized time history has a closed-form mathematical formula. The corresponding wavelet table may also be useful for identifying structural modal frequencies.

The following concerns will be addressed in future research:

1. The brute force method can be made more efficient using convergence algorithms, perhaps drawing from the field of genetics.

2. Optimization could be performed to reduce the peak displacement while still meeting the goals of "acceleration time history resemblance" and SRS fulfillment.

3. The test fixture may have different mechanical impedance than the actual mounting surface in the vehicle.

References

1. NASA-HDBK-7005, Dynamic Environmental Criteria, 2001.
2. D. Smallwood, Shock Testing on Shakers by Using Digital Control, Technology Monograph, Institute of Environmental Sciences, Mount Prospect, Illinois, 1986.
3. C. Harris, editor, Shock and Vibration Handbook, Fourth Edition; A. Piersol, Chapter 20, "Test Criteria and Specifications," McGraw-Hill, New York, 1996.
4. Himelblau, et al., "Development of Cassini Acoustics Criteria Using Titan IV Flight Data, Draft of Paper to be presented at IES ATM, May 1992.

APPENDIX A

Excerpt from Reference 1

6.6 Reconstruction of Waveforms for Transients. The maximum expected environment (MEE) for transients is commonly computed in the frequency domain using the procedures detailed in Sections 6.1 through 6.4, where the MEE represents a conservative limit for a collection of measured or predicted spectra defining the transient environment in a structural zone of concern. Either Fourier spectra, energy spectra, or shock response spectra, as defined in Sections 2.2.8 through 2.2.10, might be used to compute the MEE. On the other hand, some of the test procedures discussed in Section 10, particularly those applicable to low frequency (below 100 Hz) transient simulations on electrodynamic shakers require a time history (waveform) for the specified test signal. When the MEE is defined in terms of a shock response spectrum (SRS), there is no direct analytical way to reconstruct a representative waveform because the SRS does not have a unique relationship to the waveform from which it is computed. For this case, test time history signals with an appropriate waveform are usually constructed using decaying sine waves [6.34 - 6.36] or wavelets [6.35, 6.36] ([6.36] includes FORTRAN programs). Also, energy spectra do not lend themselves to waveform reconstruction because they have no phase information. It follows that Fourier spectra should be used to define the MEE for transients when the reconstruction of a waveform is required for test simulation purposes.

APPENDIX B

Wavelet Table for the First Example

Accel(G)	Freq(Hz)	NHS	Delay(sec)
31.42	74.59	9	0.0119
24.41	80.76	17	0.0232
-23.19	44.74	3	0.0128
20.38	149.91	7	0.0159
-14.22	41.27	7	0.1072
-19.50	63.18	5	0.0285
-10.55	124.46	19	0.0242
-5.60	83.75	19	0.0306
9.76	55.67	3	0.0086
6.28	73.98	9	0.1353
-4.04	38.95	13	0.0125
6.99	153.87	11	0.0163
4.24	55.43	13	0.0789
-9.02	98.88	5	0.0902
10.44	168.46	7	0.0591
-3.79	16.36	5	0.0048
12.22	325.26	9	0.0620
-12.66	426.15	3	0.0332
-9.35	360.30	19	0.0777
-10.63	609.62	13	0.0257
-3.96	97.17	9	0.1520
-6.34	157.05	17	0.1399
5.93	230.17	17	0.0250
-4.00	57.68	5	0.0160
-4.88	153.39	15	0.0776
-1.58	26.43	9	0.0166
7.24	434.76	13	0.0253
8.64	627.99	9	0.0209
-7.81	335.76	7	0.1347
8.85	345.89	7	0.1126

Accel(G)	Freq(Hz)	NHS	Delay(sec)
4.64	593.45	27	0.0418
-8.86	273.93	5	0.0826
-4.29	102.91	3	0.1354
4.87	532.84	23	0.0563
-4.06	93.69	7	0.0560
-5.22	314.89	15	0.0263
6.51	146.43	3	0.0478
7.61	765.69	7	0.0688
-3.78	135.14	7	0.0046
-3.38	113.53	9	0.1584
-2.75	91.92	9	0.1496
3.37	469.06	21	0.0430
5.85	698.73	13	0.0862
-5.34	865.48	13	0.0306
5.38	383.49	11	0.1214
-1.89	19.78	5	0.0010
-6.43	1034.55	9	0.0936
-6.26	312.02	3	0.1454
-2.20	133.06	21	0.0495
-4.42	290.23	7	0.1564
-4.39	1343.20	21	0.0406
5.74	992.92	11	0.0784
2.83	282.84	11	0.0664
1.48	49.99	17	0.0166
2.29	263.19	13	0.0081
2.08	63.73	5	0.1576
2.33	35.77	3	0.1567
-5.70	379.76	5	0.0934
2.99	45.05	3	0.0003
4.54	1334.48	27	0.0827

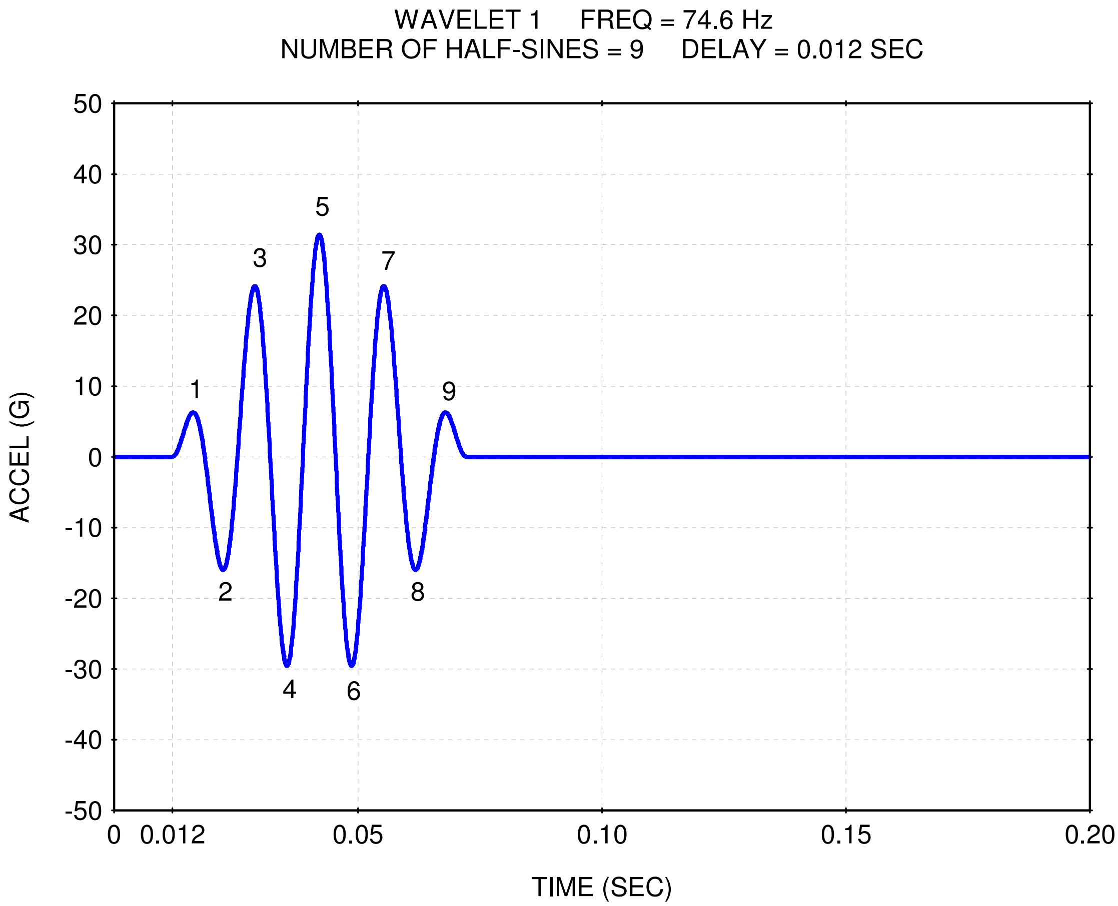

Figure B-1.

The first wavelet is shown in Figure B-1.

An equivalent acceleration formula for equation (1) is

$$W_m(t) = -\frac{A_m}{2}\cos\left[2\pi f_m\left(\frac{1}{N_m}+1\right)(t - t_{dm})\right] + \frac{A_m}{2}\cos\left[2\pi f_m\left(\frac{1}{N_m}-1\right)(t - t_{dm})\right],$$

$$\text{for } t_{dm} \le t \le \left[t_{dm} + \frac{N_m}{2f_m}\right] \tag{B-1}$$

Obviously, a given wavelet has a beat frequency effect with two spectral lines over the defined interval. The corresponding "spectral magnitude function" of the waveform in Figure (B-1) is shown in Figure (B-2).

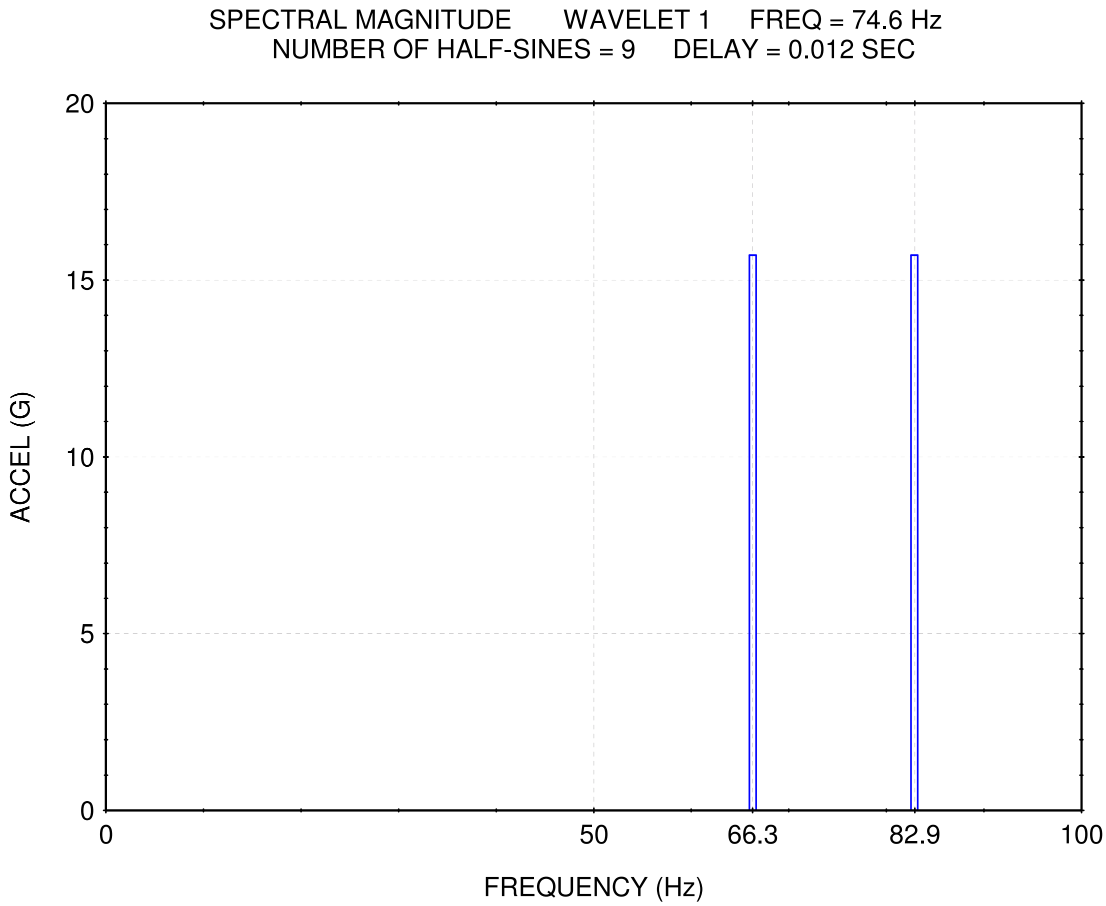

Figure B-2.

The spectral magnitude function is somewhat analogous to a Fourier transform magnitude.

An actual Fourier transform of the data would be of limited value since the energy would be smeared over several frequencies due to “leakage” and other error sources.

Note that the frequency increment of a Fourier transform is equal to the reciprocal of the signal duration. A Fourier transform is thus more suitable for data sets with longer durations.

APPENDIX C

P95/50 Method

The P95/50 rule yields the maximum predicted level, which is equal to or greater than the value at the ninety-fifth percentile at least 50% of the time.

The "95" in the P95/50 rule is taken as the 95% probability in the Normal distribution. The "50" is the 50% confidence value in the Chi-square distribution.

The tolerance value is applied to the sample standard deviation to yield an estimate of the upper limit at each frequency as follows:

$$\text{Limit} = \bar{x} + ks \qquad \text{(C-1)}$$

where $\bar{x}$ is the mean and s is the sample standard deviation.

Table C-1. P95/50 Tolerance Factors

Number of Samples	Tolerance Factor
2	2.339
3	1.939
4	1.830
5	1.779
6	1.750
7	1.732
8	1.719
9	1.709
10	1.702
11	1.696
12	1.691
13	1.687
14	1.684
15	1.681

Number of Samples	Tolerance Factor
16	1.678
17	1.676
18	1.674
19	1.673
20	1.671
21	1.670
22	1.669
23	1.668
24	1.667
25	1.666
30	1.662
35	1.659
40	1.658
∞	1.64485

APPENDIX D

Creating a Composite Pulse

The composite pulse is the sum of several individual signals. The raw signals cannot be simply added together, however. There are several concerns that must be addressed so that a reasonable sum is achieved.

Accelerometer Mounting

Consider a pair of accelerometers mounting in the same axis. The accelerometers may or may not be mounted with the same polarity. In other words, one accelerometer may be mounted in the positive axis, and the other in the negative axis. If so, the signal from one accelerometer would be inverted with respect to the other signal assuming that the waveform is simultaneous and in-phase at each location.

Note that in some cases, the accelerometer mounting diagram may not be readily available to the engineer who is reducing the measured data.

Field Types

Another concern arises from the distance of the accelerometers with respect to the source location, as well as the distance between the accelerometers themselves. Some shock events such as pyrotechnic stage separation may have a well-defined source location. Other events such as water impact may have a complex, distributed source. Regardless, the source shock has the potential of generating both traveling and standing waves. The standing waves represent modes.

The “near field” response to a discrete shock source is dominated by waves. The “far field” response is largely due to structural modes. The “mid field” response is a combination of each type.

Response to Traveling Waves

Consider a wave-like response that is measured at two accelerometer locations. There may be a measurable time delay between the two responses if the accelerometers are mounted sufficiently apart from one another. Obviously, the speed of sound in the material enters into this calculation. Furthermore, the wave speed varies depending on the wave type. The wave speed may even vary with frequency, as is the case with traveling bending waves. Similarly, dispersion may occur.

Modal Response

There are three main scenarios for two accelerometers measuring a given vibration mode.

1. The accelerometers may be in-phase.
2. The accelerometers may be 180 degrees out-of-phase.
3. Either accelerometer may be on a nodal line.

Either destructive or constructive interference may result from adding the signals.

Furthermore, the magnitude of the response at each location may vary regardless of phase.

Filtering

The telemetry data may have been filtered in some manner that introduces a phase delay.

Synchronization

Accelerometer data from more than one flight may be available. The data may or may not be synchronized to a common starting time. The "true starting time" may even be a matter of engineering judgment.

Summation

As a result of these concerns, there is no "exact" method for summing accelerometer signals for the purpose of deriving a composite pulse.

Again, brute force random number generation may be used.

One step is to multiply each signal by either +1 or -1. A second step is to shift each signal by some "small" time delay. Each of these steps is performed in a random manner over hundreds of trials.

The final composite pulse is the one which yields the greatest RMS value.

A possible concern is that this approach may emphasize certain modal frequency while attenuating others.

Again, the wavelet components are re-scaled to meet the P95/50 SRS, as explained in the main text. Thus, all frequency components should be represented to the proper amplitude in the final wavelet series.

APPENDIX E

Wavelet Velocity

Again, the equation for an individual wavelet acceleration is

$$W_m(t) = \begin{cases} 0, \quad \text{for } t < t_{dm} \\ A_m \sin\left[\frac{2\pi f_m}{N_m}(t - t_{dm})\right] \sin[2\pi f_m (t - t_{dm})], \quad \text{for } t_{dm} \le t \le \left[t_{dm} + \frac{N_m}{2f_m}\right] \\ 0, \quad \text{for } t > \left[t_{dm} + \frac{N_m}{2f_m}\right] \end{cases}$$

(E-1a)

The velocity $V_m(t)$ at time t is

$$V_m(t) = \int_{t_{dm}}^{t} A_m \sin\left[\frac{2\pi f_m}{N_m}(\tau - t_{dm})\right] \sin[2\pi f_m (\tau - t_{dm})] d\tau,$$

$$\text{for } t_{dm} \le t \le \left[t_{dm} + \frac{N_m}{2f_m}\right]$$

(E-2)

Let

$$u = \tau - t_{dm}$$

$$du = d\tau$$

$$\alpha = \frac{2\pi f_m}{N_m}$$

$$\beta = 2\pi f_m$$

$$V_m(t) = A_m \int_{u_1}^{u_2} \sin[\alpha u] \sin[\beta u] du \tag{E-3}$$

$$V_m(t) = -\frac{1}{2} A_m \int_{u_1}^{u_2} \cos[(\alpha+\beta)u]\, du \quad + \quad \frac{1}{2} A_m \int_{u_1}^{u_2} \cos[(\alpha-\beta)u] du \tag{E-4}$$

$$V_m(t) = -\frac{A_m}{2(\alpha+\beta)} \sin[(\alpha+\beta)u]\Big|_{u_1}^{u_2} + \frac{A_m}{2(\alpha-\beta)} \sin[(\alpha-\beta)u]\Big|_{u_1}^{u_2} \tag{E-5}$$

$$V_m(t) = -\frac{A_m}{2(\alpha+\beta)} \sin[(\alpha+\beta)(\tau - t_{dm})]\Big|_{t_{dm}}^{t} + \frac{A_m}{2(\alpha-\beta)} \sin[(\alpha-\beta)(\tau - t_{dm})]\Big|_{t_{dm}}^{t} \tag{E-6}$$

The wavelet velocity equation is thus

$$V_m(t) = -\frac{A_m}{2(\alpha+\beta)} \sin[(\alpha+\beta)(t - t_{dm})] + \frac{A_m}{2(\alpha-\beta)} \sin[(\alpha-\beta)(t - t_{dm})],$$

$$\text{for } t_{dm} \le t \le \left[t_{dm} + \frac{N_m}{2f_m} \right] \tag{E-7}$$

The wavelet ends at $$t = \left[t_{dm} + \frac{N_m}{2f_m} \right] = \left[t_{dm} + \frac{\pi}{\alpha} \right]$$

The velocity at the end time is

$$V_m\left(t_{dm} + \frac{\pi}{\alpha} \right) = -\frac{A_m}{2(\alpha+\beta)} \sin\left[(\alpha+\beta)\left(\left(t_{dm} + \frac{\pi}{\alpha} \right) - t_{dm} \right) \right]$$
$$+ \frac{A_m}{2(\alpha-\beta)} \sin\left[(\alpha-\beta)\left(\left(t_{dm} + \frac{\pi}{\alpha} \right) - t_{dm} \right) \right] \tag{E-8}$$

$$V_m\left(t_{dm} + \frac{\pi}{\alpha} \right) = -\frac{A_m}{2(\alpha+\beta)} \sin\left[\left(1 + \frac{\beta}{\alpha} \right)\pi \right] + \frac{A_m}{2(\alpha-\beta)} \sin\left[\left(1 - \frac{\beta}{\alpha} \right)\pi \right] \tag{E-9}$$

Note that

$$\frac{\beta}{\alpha} = N_m, \quad \text{an odd integer} \geq 3$$

Thus

$$\sin\left[\left(1 + \frac{\beta}{\alpha} \right)\pi \right] = 0 \tag{E-10}$$

$$\sin\left[\left(1 - \frac{\beta}{\alpha} \right)\pi \right] = 0 \tag{E-11}$$

The net velocity is thus

$$V_m\left(t_{dm} + \frac{\pi}{\alpha} \right) = 0 \tag{E-12}$$

APPENDIX F

Wavelet Displacement

The wavelet displacement $D_m(t)$ for wavelet m is obtained by integrating the velocity.

$$D_m(t) = -\frac{A_m}{2(\alpha+\beta)}\int_{t_{dm}}^{t} \sin[(\alpha+\beta)(\tau - t_{dm})]d\tau$$

$$+\frac{A_m}{2(\alpha-\beta)}\int_{t_{dm}}^{t} \sin[(\alpha-\beta)(\tau - t_{dm})]d\tau,$$

$$\text{for } t_{dm} \le t \le \left[t_{dm} + \frac{N_m}{2f_m}\right]$$

(F-1)

Again,

$$\alpha = \frac{2\pi f_m}{N_m}$$

$$\beta = 2\pi f_m$$

$$D_m(t) = +\frac{A_m}{2(\alpha+\beta)^2}\cos[(\alpha+\beta)(\tau - t_{dm})]\Big|_{t_{dm}}^{t}$$

$$-\frac{A_m}{2(\alpha-\beta)^2}\cos[(\alpha-\beta)(\tau - t_{dm})]\Big|_{t_{dm}}^{t}$$

(F-2)

The displacement equation is thus

$$D_m(t) = +\frac{A_m}{2(\alpha+\beta)^2}\{\cos[(\alpha+\beta)(t-t_{dm})]-1\}-\frac{A_m}{2(\alpha-\beta)^2}\{\cos[(\alpha-\beta)(t-t_{dm})]-1\},$$

$$\text{for } t_{dm} \le t \le \left[t_{dm}+\frac{N_m}{2f_m}\right]$$

(F-3)

The wavelet ends at $t = \left[t_{dm}+\frac{N_m}{2f_m}\right] = \left[t_{dm}+\frac{\pi}{\alpha}\right]$

The final displacement is thus

$$D_m\left(t_{dm}+\frac{\pi}{\alpha}\right) = +\frac{A_m}{2(\alpha+\beta)^2}\left\{\cos\left[\left(1+\frac{\beta}{\alpha}\right)\pi\right]-1\right\}-\frac{A_m}{2(\alpha-\beta)^2}\left\{\cos\left[\left(1-\frac{\beta}{\alpha}\right)\pi\right]-1\right\}$$

(F-4)

Note that

$$\frac{\beta}{\alpha} = N_m, \quad \text{an odd integer} \ge 3$$

Thus

$$\cos\left[\left(1+\frac{\beta}{\alpha}\right)\pi\right]-1 = 0 \tag{F-5}$$

$$\cos\left[\left(1-\frac{\beta}{\alpha}\right)\pi\right]-1 = 0 \tag{F-6}$$

The net displacement is thus

$$D_m\left(t_{dm}+\frac{\pi}{\alpha}\right) = 0 \tag{F-7}$$

APPENDIX G

Software Programs

Program	Description
wavelet_reconstruct.cpp	Synthesizes a wavelet series to represent a measured time history.
composite_shock.cpp	Adds multiple waveforms using inversion and time delays to maximize the RMS.
srs_9550.cpp	Calculates the P95/50 level for two or more shock response spectra.
wavelet_scale_SRS.cpp	Scales the individual wavelets of a series so that the resulting time history satisfies an SRS specification.
th_from_wavelet_table.cpp	Generates a time history from a wavelet table.
qsrs.cpp	Calculates an SRS for an acceleration time history.

The program files are included in the initial submission of this paper to the customer. Other interested parties may contact the author for copies of the code.

The programs are DOS or "console mode." Also, the programs are nearly straight C rather than C++.

The programs were written using Microsoft Visual C++ 6.0. The programs do not use any GUIs or Visual features, however. Some minor changes may be required for other compilers.