RESPONSE OF A SINGLE-DEGREE-OF-FREEDOM SYSTEM SUBJECTED TO A UNIT STEP DISPLACEMENT

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Introduction

Consider the single-degree-of-freedom system in Figure 1.

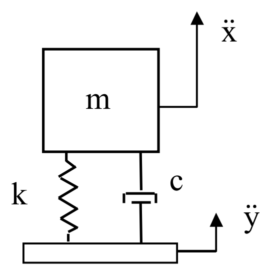

Figure 1.

where

m is the mass
c is the viscous damping coefficient
k is the stiffness
x is the absolute displacement of the mass
y is the base input displacement

A free-body diagram is shown in Figure 2.

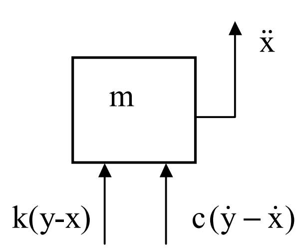

Figure 2.

Summation of forces in the vertical direction

$$\sum F = m\ddot{x} \tag{1}$$

$$m\ddot{x} = c(\dot{y} - \dot{x}) + k(y - x) \tag{2}$$

$$m\ddot{x} + c\dot{x} + kx = c\dot{y} + ky \tag{3}$$

$$\ddot{x} + (c/m)\dot{x} + (k/m)x = (c/m)\dot{y} + (k/m)y \tag{4}$$

$$(c/m) = 2\xi\omega_n \tag{5}$$

$$(k/m) = \omega_n^2 \tag{6}$$

where ω_n is the natural frequency in (radians/sec), and ξ is the damping ratio.

$$\ddot{x} + 2\xi\omega_n\dot{x} + \omega_n^2 x = 2\xi\omega_n\dot{y} + \omega_n^2 y \tag{7}$$

The base input displacement y is the unit step function u(t). The base input velocity is the Dirac delta function $\delta(t)$.

Multiply the base input displacement by a scale factor D for a more general solution, even though the goal is to solve for a unit impulse.

$$\ddot{x} + 2\xi\omega_n\dot{x} + \omega_n^2 x = 2\xi\omega_n D\delta(t) + \omega_n^2 Du(t) \tag{8}$$

Now take the Laplace transform.

$$L\left\{\ddot{x} + 2\xi\omega_n\dot{x} + \omega_n^2 x\right\} = L\left\{2\xi\omega_n D\delta(t) + \omega_n^2 Du(t)\right\} \tag{9}$$

$$\begin{aligned} &s^2X(s) - sx(0) - \dot{x}(0) \\ &+ 2\xi\omega_n sX(s) - 2\xi\omega_n x(0) \\ &+ \omega_n^2 X(s) = 2\xi\omega_n D + \frac{\omega_n^2}{s}D \end{aligned} \tag{10}$$

$$\left\{s^2 + 2\xi\omega_n s + \omega_n^2\right\}X(s) + \{-1\}\dot{x}(0) + \{-s - 2\xi\omega_n\}x(0) = 2\xi\omega_n D + \frac{\omega_n^2}{s}D \tag{11}$$

$$\left\{s^2 + 2\xi\omega_n s + \omega_n^2\right\}X(s) = \dot{x}(0) + \{s + 2\xi\omega_n\}x(0) + 2\xi\omega_n D + \frac{\omega_n^2}{s}D \tag{12}$$

$$\left\{s^2 + 2\xi\omega_n s + \omega_n^2\right\}X(s) = s\,x(0) + \dot{x}(0) + 2\xi\omega_n[x(0) + D] + \frac{\omega_n^2}{s}D \tag{13}$$

$$X(s) = \frac{s\,x(0)}{\left\{s^2 + 2\xi\omega_n s + \omega_n^2\right\}} + \frac{\dot{x}(0) + 2\xi\omega_n[x(0) + D]}{\left\{s^2 + 2\xi\omega_n s + \omega_n^2\right\}} + \frac{\omega_n^2 D}{s\left\{s^2 + 2\xi\omega_n s + \omega_n^2\right\}} \tag{14}$$

Consider the denominator term,

$$s^2 + 2\xi\omega_n s + \omega_n^2 = \left(s + \xi\omega_n\right)^2 + \omega_n^2 - \left(\xi\omega_n\right)^2 \tag{15}$$

$$s^2 + 2\xi\omega_n s + \omega_n^2 = \left(s + \xi\omega_n\right)^2 + \omega_n^2\left(1 - \xi^2\right) \tag{16}$$

Now define the damped natural frequency,

$$\omega_d = \omega_n\sqrt{1 - \xi^2} \tag{17}$$

Substitute equation (17) into (16),

$$s^2 + 2\xi\omega_n s + \omega_n^2 = (s + \xi\omega_n)^2 + \omega_d^2 \tag{18}$$

Substitute equation (18) into (14).

$$X(s) = \frac{s\,x(0)}{(s+\xi\omega_n)^2 + \omega_d^2} + \frac{\dot{x}(0) + 2\xi\omega_n[x(0)+D]}{(s+\xi\omega_n)^2 + \omega_d^2} + \frac{\omega_n^2 D}{s\left\{(s+\xi\omega_n)^2 + \omega_d^2\right\}} \tag{19}$$

$$X(s) = \frac{(s+\xi\omega_n)x(0)}{(s+\xi\omega_n)^2 + \omega_d^2} + \frac{\dot{x}(0) + \xi\omega_n x(0) + 2\xi\omega_n D}{(s+\xi\omega_n)^2 + \omega_d^2} + \frac{\omega_n^2 D}{s\left\{(s+\xi\omega_n)^2 + \omega_d^2\right\}} \tag{20}$$

Perform partial fraction expansion per Appendix A.

$$\frac{1}{s\left\{(s+\xi\omega_n)^2 + \omega_d^2\right\}} = \left(\frac{1}{\omega_n^2}\right)\frac{1}{s} + \left(\frac{-1}{\omega_n^2}\right)\frac{(s+2\xi\omega_n)}{(s+\xi\omega_n)^2 + \omega_d^2} \tag{21}$$

$$X(s) = \frac{(s+\xi\omega_n)x(0)}{(s+\xi\omega_n)^2 + \omega_d^2} + \frac{\dot{x}(0) + \xi\omega_n x(0) + 2\xi\omega_n D}{(s+\xi\omega_n)^2 + \omega_d^2}$$
$$+\left(\frac{\omega_n^2 D}{\omega_n^2}\right)\frac{1}{s} + \left(\frac{-\omega_n^2 D}{\omega_n^2}\right)\frac{(s+2\xi\omega_n)}{(s+\xi\omega_n)^2 + \omega_d^2} \tag{22}$$

$$X(s) = \frac{(s+\xi\omega_n)x(0)}{(s+\xi\omega_n)^2 + \omega_d^2} + \frac{\dot{x}(0) + \xi\omega_n x(0)}{(s+\xi\omega_n)^2 + \omega_d^2}$$
$$+\frac{1}{s}D + \frac{-Ds}{(s+\xi\omega_n)^2 + \omega_d^2} \tag{23}$$

Take the inverse Laplace transform using standard tables.

$$X(s) = \frac{(s+\xi\omega_n)x(0)}{(s+\xi\omega_n)^2+\omega_d{}^2} + \frac{\dot{x}(0)+\xi\omega_n x(0)}{(s+\xi\omega_n)^2+\omega_d{}^2}$$

$$+\frac{1}{s}D + \frac{-Ds}{(s+\xi\omega_n)^2+\omega_d{}^2} \tag{24}$$

$$x(t) =$$

$$+\exp(-\xi\omega_n t)\left\{[\xi\omega_n x(0)]\cos(\omega_d t) + \left[\frac{\dot{x}(0)+\xi\omega_n x(0)}{\omega_d}\right]\sin(\omega_d t)\right\}$$

$$+ D\,u(t)$$

$$- D\exp(-\xi\omega_n t)\left\{\cos(\omega_d t) + \left[\frac{-\xi\omega_n}{\omega_d}\right]\sin(\omega_d t)\right\} \tag{25}$$

$$x(t) =$$

$$+\exp(-\xi\omega_n t)\left\{[\xi\omega_n x(0)]\cos(\omega_d t) + \left[\frac{\dot{x}(0)+\xi\omega_n x(0)}{\omega_d}\right]\sin(\omega_d t)\right\}$$

$$+ D\left\{u(t) - \exp(-\xi\omega_n t)\left\{\cos(\omega_d t) + \left[\frac{-\xi\omega_n}{\omega_d}\right]\sin(\omega_d t)\right\}\right\} \tag{26}$$

Example

Consider a single-degree-of-freedom system with a natural frequency of 100 Hz and 5% damping. The system has zero initial velocity and zero initial displacement. The system is subjected to a unit displacement at $t = 0^+$, which is multiplied by a coefficient of D = 1 mm. The response is shown in Figure 3.

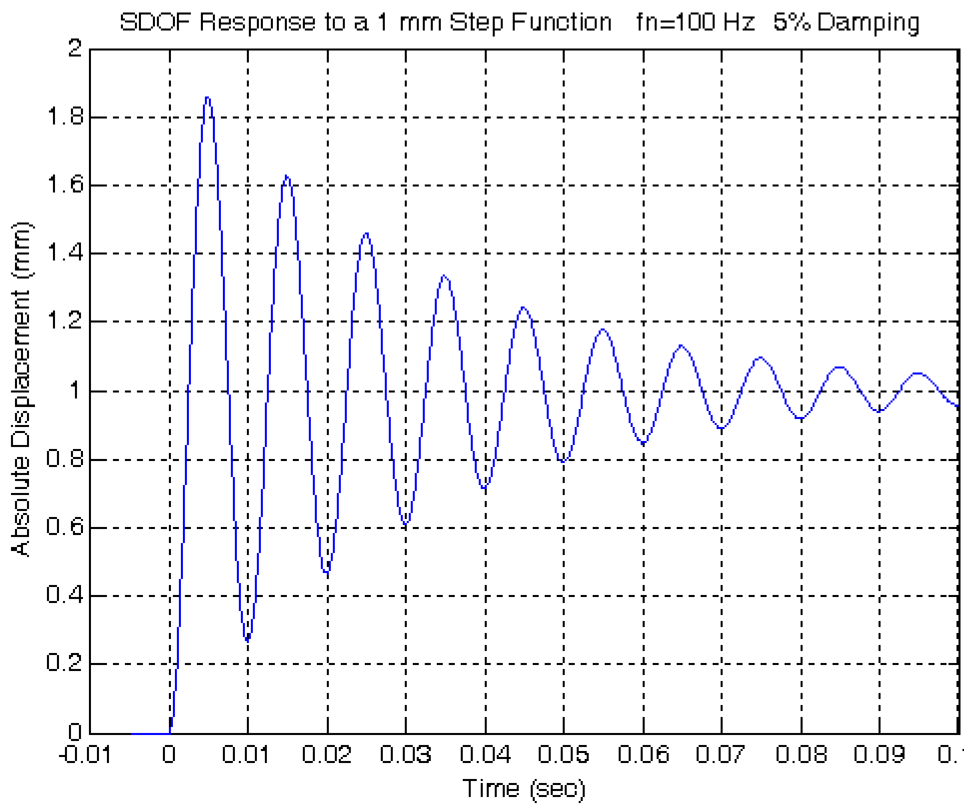

Figure 3.

APPENDIX A

Partial Fraction Expansion

$$\frac{1}{s\left\{(s+\xi\omega_n)^2+\omega_d{}^2\right\}} = \frac{A}{s} + \frac{Bs+C}{(s+\xi\omega_n)^2+\omega_d{}^2} \qquad \text{(A-1)}$$

$$\frac{A}{s}s\left\{(s+\xi\omega_n)^2+\omega_d{}^2\right\} + \frac{Bs+C}{(s+\xi\omega_n)^2+\omega_d{}^2}s\left\{(s+\xi\omega_n)^2+\omega_d{}^2\right\}=1 \qquad \text{(A-2)}$$

$$A\left\{(s+\xi\omega_n)^2+\omega_d{}^2\right\} + Bs^2+Cs=1 \qquad \text{(A-3)}$$

$$s^2+2\xi\omega_n s+\omega_n{}^2=(s+\xi\omega_n)^2+\omega_d{}^2 \qquad \text{(A-4)}$$

$$A\left\{s^2+2\xi\omega_n s+\omega_n{}^2\right\}+ Bs^2+Cs=1 \qquad \text{(A-5)}$$

$$\{A+B\}s^2+\{2\xi\omega_n A+C\}s+A\omega_n{}^2=1 \qquad \text{(A-6)}$$

$$\{A+B\}s^2=0 \qquad \text{(A-7)}$$

$$\{2\xi\omega_n A+C\}s=0 \qquad \text{(A-8)}$$

$$A\omega_n{}^2=1 \qquad \text{(A-9)}$$

$$A=\frac{1}{\omega_n{}^2} \qquad \text{(A-10)}$$

$$B=\frac{-1}{\omega_n{}^2} \qquad \text{(A-11)}$$

$$C = -2\xi\omega_n A \tag{A-12}$$

$$C = \frac{-2\xi}{\omega_n} \tag{A-13}$$

$$\frac{1}{s\left\{(s+\xi\omega_n)^2 + \omega_d^{\,2}\right\}} = \left(\frac{1}{\omega_n^{\,2}}\right)\frac{1}{s} + \frac{\left(\frac{-1}{\omega_n^{\,2}}\right)s + \left(\frac{-2\xi}{\omega_n}\right)}{(s+\xi\omega_n)^2 + \omega_d^{\,2}} \tag{A-14}$$

$$\frac{1}{s\left\{(s+\xi\omega_n)^2 + \omega_d^{\,2}\right\}} = \left(\frac{1}{\omega_n^{\,2}}\right)\frac{1}{s} + \left(\frac{-1}{\omega_n^{\,2}}\right)\frac{(s+2\xi\omega_n)}{(s+\xi\omega_n)^2 + \omega_d^{\,2}} \tag{A-15}$$

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