Boundary Element Method: Acoustic Radiation and Scattering

bem_vibroacoustics_a ver 1.2 by Tom Irvine solver: bem_vibro_engine.m

1. Direct boundary integral equation (time factor $e^{j\omega t}$, $\mathbf{n}$ = outward normal of the body)

Helmholtz equation	$\nabla^2 p + k^2 p = 0, \qquad k = \omega/c, \quad \omega = 2\pi f$
Free-space Green's function	$G(\mathbf{x},\mathbf{y}) = \dfrac{e^{-jkr}}{4\pi r}, \qquad r = \lvert\mathbf{x}-\mathbf{y}\rvert$
Neumann data (vibrating surface)	$q = \dfrac{\partial p}{\partial n} = -j\omega\rho\, v_n$
Integral equation	$c(\mathbf{x})\, p(\mathbf{x}) = \int_S \left[p\, \dfrac{\partial G}{\partial n_y} - G\, q \right] dS, \qquad c = 1$ in the fluid, $c = \dfrac{1}{2}$ on a smooth surface
Constant elements, centroid collocation	$\left(\dfrac{1}{2}\mathbf{I} - \mathbf{H}\right)\mathbf{p} = -\mathbf{G}\,\mathbf{q}, \qquad H_{ij} = \dfrac{\partial G}{\partial n}(\mathbf{x}_i,\mathbf{y}_j)\, A_j, \quad G_{ij} = G(\mathbf{x}_i,\mathbf{y}_j)\, A_j$
Singular self terms (equal-area disk)	$G_{ii} = \dfrac{a_{eq}}{2} - \dfrac{jkA_i}{4\pi}, \qquad a_{eq} = \sqrt{A_i/\pi}, \qquad H_{ii} = 0$ (carried by the $\dfrac{1}{2}\mathbf{I}$ term)
Field points	$p(\mathbf{x}) = \sum_j \left[\dfrac{\partial G}{\partial n}(\mathbf{x},\mathbf{y}_j)\, p_j - G(\mathbf{x},\mathbf{y}_j)\, q_j \right] A_j$
CHIEF rows at interior points $\mathbf{x}_c$	$\sum_j \left[H_{cj}\, p_j - G_{cj}\, q_j \right] = 0$; the over-determined system is solved by least squares
Rigid ground plane $z = 0$	$G_h = G(\mathbf{x},\mathbf{y}) + G(\mathbf{x},\mathbf{y}'), \qquad \mathbf{y}' = (y_1, y_2, -y_3)$, normal mirrored the same way
Rigid scatterer, incident plane wave	$p_{inc} = p_0\, e^{-jkz}, \qquad q_s = -\dfrac{\partial p_{inc}}{\partial n}, \qquad p_{total} = p_{inc} + p_s$

2. Results

Sound power and level	$W = \dfrac{1}{2}\,\mathrm{Re}\sum_j p_j\, v_{n,j}^*\, A_j, \qquad L_w = 10\log_{10}\dfrac{W}{10^{-12}\ \mathrm{W}}$
Radiation efficiency	$\sigma = W \Big/ \left(\dfrac{1}{2}\rho c \sum_j \lvert v_{n,j}\rvert^2 A_j \right)$
Sound pressure level	$L_p = 20\log_{10}\dfrac{\lvert p\rvert}{\sqrt{2}\, p_{ref}}, \qquad p_{ref} = 20\ \mu$Pa (air), $1\ \mu$Pa (water); amplitudes are peak
Far-field amplitude	$p \to F(\hat{\mathbf{r}})\dfrac{e^{-jkR}}{R}, \qquad F = \dfrac{1}{4\pi}\sum_j \left[jk(\hat{\mathbf{r}}\cdot\mathbf{n}_j)\, p_j - q_j \right] e^{jk\hat{\mathbf{r}}\cdot\mathbf{y}_j} A_j$
Backscatter target strength	$TS = 20\log_{10}\dfrac{\lvert F(\theta = 180^\circ)\rvert}{p_0 \cdot 1\ \mathrm{m}}$ (incident wave travels along $+z$)
Compact monopole (low frequency)	$W = \dfrac{\rho c k^2 \lvert Q\rvert^2}{8\pi}, \qquad Q = \sum_j v_{n,j} A_j$

3. Irregular frequencies and mesh

First interior Dirichlet frequency	sphere: $ka = \pi$; box: $f = \dfrac{c}{2}\sqrt{L_x^{-2} + L_y^{-2} + L_z^{-2}}$; cylinder: $f = \dfrac{c}{2\pi}\sqrt{(2.405/R)^2 + (\pi/L)^2}$
Mesh density	at least 6 (preferably 8 to 10) elements per wavelength $\lambda = c/f$ at the highest frequency
Icosphere	$N = 20 \cdot 4^n$ triangles ($n = 3$: 1280, $n = 4$: 5120); dense matrix storage $16N^2$ bytes

The plain direct BIE has no unique solution at the interior frequencies (a mathematical artifact, not a resonance). CHIEF adds interior points as extra equations. Example: a pulsating sphere sweep through $ka = \pi$ has a spike of about 12 dB in L_w without CHIEF.

Boundary Element Method (page 2)

4. Closed-form checks ($h_n = h_n^{(2)} = j_n - j\,y_n$, outgoing for $e^{j\omega t}$; P_n = Legendre polynomial)

Sphere, $v_n = \sum_n U_n P_n(\cos\theta)$	$p(r,\theta) = -j\rho c \sum_n U_n \dfrac{h_n(kr)}{h_n'(ka)} P_n(\cos\theta)$
Sphere sound power	$W = 2\pi a^2 \sum_n \dfrac{\mathrm{Re}(Z_n)\,\lvert U_n\rvert^2}{2n+1}, \qquad Z_n = -j\rho c\,\dfrac{h_n(ka)}{h_n'(ka)}$
1. Pulsating sphere, $U_0 = U$	$p = \rho c U \dfrac{jka}{1+jka}\,\dfrac{a}{r}\,e^{-jk(r-a)}, \qquad \sigma = \dfrac{(ka)^2}{1+(ka)^2}$
2. Oscillating sphere, $U_1 = U$	$p = -j\rho c U \cos\theta\,\dfrac{h_1(kr)}{h_1'(ka)}, \qquad \sigma = \dfrac{(ka)^4}{4+(ka)^4}$
3. Rigid sphere (Mie series)	$p_s = -p_0 \sum_n (2n+1)(-j)^n \dfrac{j_n'(ka)}{h_n'(ka)}\,h_n(kr)\,P_n(\cos\theta), \qquad TS \to 20\log_{10}(a/2)$ for $ka \gg 1$
4. Cap of half-angle θ_0	$U_0 = \dfrac{U}{2}(1-\cos\theta_0), \qquad U_n = \dfrac{U}{2}\left[P_{n-1}(\cos\theta_0) - P_{n+1}(\cos\theta_0)\right]$
7. Sphere at height h over rigid ground	$p \approx p_{mono}(r_1) + p_{mono}(r_2), \qquad W = W_{free}\left[1 + \dfrac{\sin 2kh}{2kh}\right]$ (image source, $ka \ll 1$)
5, 6, 8. Box, cylinder, STL	compact monopole power (section 2), low frequency only

At $ka = 1$: $\sigma = 0.5$ (pulsating), 0.2 (oscillating); $W_{osc}/W_{puls} = 0.2/1.5 = 0.133$ for the same U. σ uses the definition in section 2.

5. Problems in the app

1 to 3: slide problems (pulsating sphere, oscillating sphere, rigid sphere scattering), checked against the closed forms above.

4: cap on a rigid sphere: loudspeaker in a spherical cabinet or transducer face. Partly covered elements get the covered area fraction.

5: box housing (gearbox, pump casing, oil pan): top-face piston, top plate driven by a point force (page 3), all faces breathing, or rigid-body translation along z.

6: closed cylinder (tank, motor housing, pipe, sonar housing): side wall breathing, end cap piston, rigid-body translation.

7: pulsating sphere above a rigid floor (half-space Green's function): ground-reflection interference lobes.

8: any closed STL surface: uniform pulsating, rigid-body translation, or plane-wave scattering. Sample: bem_sample_housing.stl.

Units

metric: m, m/s, kg/m^3, m/s, Pa. **English**: in, in/s, lbm/ft^3, ft/s, psi, converted to SI internally (1 in = 0.0254 m, 1 lbm/ft^3 = 16.018 kg/m^3, 1 ft = 0.3048 m, 1 psi = 6894.76 Pa). Sound power is in W. STL coordinates times the scale factor give meters.

References

1. T. Irvine, *The Boundary Element Method for Vibroacoustics* (class slides): problems 1 to 3, bem_acoustic_solve.m.
2. P. M. Morse and K. U. Ingard, *Theoretical Acoustics*, McGraw-Hill, 1968: radiation from spheres and the spherical cap, rigid sphere scattering.
3. H. A. Schenck, "Improved integral formulation for acoustic radiation problems," *J. Acoust. Soc. Am.* 44, 1968: CHIEF.
4. A. D. Pierce, *Acoustics: An Introduction to Its Physical Principles and Applications*: Green's functions, image sources.

6. Box top plate driven by a point force (structure-to-BEM, one-way coupling)

Bending stiffness, mass per area	$D = \dfrac{Eh^3}{12(1-\nu^2)}, \qquad \mu = \rho_s h$
Natural frequencies (simply supported)	$\omega_{mn} = \pi^2 \left(\dfrac{m^2}{L_x^2} + \dfrac{n^2}{L_y^2} \right) \sqrt{\dfrac{D}{\mu}}$
Mode shapes, modal mass	$\phi_{mn} = \sin \dfrac{m\pi x}{L_x} \, \sin \dfrac{n\pi y}{L_y}, \qquad M_{mn} = \dfrac{\mu L_x L_y}{4}$
Velocity, force F at (x_0, y_0)	$v(x,y) = j\omega \sum_{m,n} \dfrac{F \, \phi_{mn}(x_0,y_0) \, \phi_{mn}(x,y)}{M_{mn} \left[\omega_{mn}^2 (1 + j\eta) - \omega^2 \right]}$, modes with $\omega_{mn} \le \max(4\omega, 4\omega_{11})$
Space-average mean-square velocity	$\langle \overline{v^2} \rangle = \dfrac{1}{8} \sum_{m,n} \lvert v_{mn} \rvert^2$ (exact, from orthogonality; compared with the element average)
Critical frequency, bending wavenumber	$f_c = \dfrac{c^2}{2\pi} \sqrt{\dfrac{\mu}{D}}, \qquad k_b = \omega^{1/2} \left(\dfrac{\mu}{D} \right)^{1/4}$
Element size	$\le \min(\lambda, \lambda_b)/n_{epw}, \qquad \lambda_b = 2\pi/k_b$
Baffled plate reference (Rayleigh)	$p(\mathbf{x}) = j\omega\rho \int_S v \, \dfrac{e^{-jkr}}{2\pi r} \, dS$, self term $\dfrac{1 - e^{-jka_{eq}}}{jk}$; W and σ as in section 2
Above f_c (Maidanik)	$\sigma \approx \left(1 - \dfrac{f_c}{f} \right)^{-1/2}$

The plate velocity is the BEM Neumann data on the top face; the other five faces are rigid. Fluid loading on the plate is neglected (light fluid), so use air, not water, for this drive. Plate natural frequencies inside the band are added to the sweep so the resonance peaks are not missed.

Below f_c the bending waves are subsonic: neighbouring regions cancel and $\sigma \ll 1$. The box in free field radiates less than the same plate in an infinite baffle, because sound also flows around the box edges. Above f_c both approach the theory curve.

Default case: 10 mm steel plate, 0.6 m × 0.4 m, on a 0.3 m tall box, $F = 10$ N, $\eta = 0.02$: $f_{11} = 217$ Hz, $f_c = 1226$ Hz.

References for section 6

5. A. W. Leissa, *Vibration of Plates*, NASA SP-160, 1969: simply supported plate frequencies and modes.
6. G. Maidanik, "Response of ribbed panels to reverberant acoustic fields," *J. Acoust. Soc. Am.* 34, 1962: radiation efficiency.
7. C. E. Wallace, "Radiation resistance of a rectangular panel," *J. Acoust. Soc. Am.* 51, 1972: baffled panel modes.