## Composite Laminate Stress, Equations

Reference: T. Irvine, Composite Material Stress, Chapters 8, 9 and 24. Plane stress.

## Micromechanics, Rule of Mixtures (Section 8.2)

Longitudinal modulus, Voigt (iso-strain) upper bound

$$E_1 = E_f V_f + E_m V_m, \qquad V_m = 1 - V_f$$

Transverse and in-plane shear moduli, Reuss (iso-stress) lower bound

$$\frac{1}{E_2} = \frac{V_f}{E_f} + \frac{V_m}{E_m}, \qquad \frac{1}{G_{12}} = \frac{V_f}{G_f} + \frac{V_m}{G_m}$$

Major Poisson ratio (rule of mixtures)

$$\nu_{12} = \nu_f V_f + \nu_m V_m, \qquad \nu_{21} = \nu_{12}\,\frac{E_2}{E_1}$$

## Lamina Reduced Stiffness (Section 9.2)

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix}$$

$$Q_{11} = \frac{E_1}{1 - \nu_{12}\nu_{21}}, \quad Q_{22} = \frac{E_2}{1 - \nu_{12}\nu_{21}}, \quad Q_{12} = \frac{\nu_{12}E_2}{1 - \nu_{12}\nu_{21}}, \quad Q_{66} = G_{12}$$

## Coordinate Transformation (Section 9.3)

Ply angle $\theta$ from the global x-axis to the fiber 1-axis, $c = \cos\theta$, $s = \sin\theta$

$$[T] = \begin{bmatrix} c^2 & s^2 & 2cs \\ s^2 & c^2 & -2cs \\ -cs & cs & c^2 - s^2 \end{bmatrix}, \qquad [R] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\{\sigma\}_{12} = [T]\{\sigma\}_{xy}, \qquad \{\varepsilon\}_{12} = [R][T][R]^{-1}\{\varepsilon\}_{xy}$$

$$[\bar{Q}] = [T]^{-1}[Q][R][T][R]^{-1}$$

## Classical Lamination Theory (Sections 9.1 and 9.4)

Strain at height z in ply k, from mid-plane strain and curvature

$$\{\varepsilon\}_k = \{\varepsilon^0\} + z\,\{\kappa\}$$

Ply interfaces: $z_1 = -h/2$ (bottom of ply 1), ..., $z_{N+1} = +h/2$

$$A_{ij} = \sum_{k=1}^{N} \bar{Q}_{ij}^{k}\,(z_{k+1} - z_k)$$

$$B_{ij} = \frac{1}{2}\sum_{k=1}^{N} \bar{Q}_{ij}^{k}\,(z_{k+1}^2 - z_k^2)$$

$$D_{ij} = \frac{1}{3}\sum_{k=1}^{N} \bar{Q}_{ij}^{k}\,(z_{k+1}^3 - z_k^3)$$

$$\begin{Bmatrix} N \\ M \end{Bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{Bmatrix} \varepsilon^0 \\ \kappa \end{Bmatrix} \quad \Rightarrow \quad \begin{Bmatrix} \varepsilon^0 \\ \kappa \end{Bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix}^{-1} \begin{Bmatrix} N \\ M \end{Bmatrix}$$

Ply stresses at the top and bottom of each ply

$$\{\sigma\}_{xy} = [\bar{Q}]_k\left(\{\varepsilon^0\} + z\{\kappa\}\right), \qquad \{\sigma\}_{12} = [T]_k\{\sigma\}_{xy}$$

## Effective Laminate Properties

In-plane, with $[a] = [A]^{-1}$, and flexural, with $[d] = [D]^{-1}$

$$E_x = \frac{1}{h\,a_{11}}, \quad E_y = \frac{1}{h\,a_{22}}, \quad G_{xy} = \frac{1}{h\,a_{66}}, \quad \nu_{xy} = -\frac{a_{12}}{a_{11}}$$

$$E_x^f = \frac{12}{h^3 d_{11}}, \qquad E_y^f = \frac{12}{h^3 d_{22}}$$

Symmetric lay-up: $B = 0$. Balanced lay-up: $A_{16} = A_{26} = 0$ (Section 9.5).

## Failure Criteria (Chapter 24), FI $\geq 1$ denotes ply failure

Strengths are positive magnitudes: $X_T, X_C, Y_T, Y_C$, in-plane shear $S = S_L$, transverse shear $S_T$

## Tsai-Hill (Section 24.3)

$$FI = \left(\frac{\sigma_1}{X}\right)^2 - \frac{\sigma_1\sigma_2}{X^2} + \left(\frac{\sigma_2}{Y}\right)^2 + \left(\frac{\tau_{12}}{S}\right)^2$$

$X = X_T$ if $\sigma_1 \geq 0$, else $X_C$. $Y = Y_T$ if $\sigma_2 \geq 0$, else $Y_C$.

## Tsai-Wu (Section 24.4)

$$FI = F_1\sigma_1 + F_2\sigma_2 + F_{11}\sigma_1^2 + F_{22}\sigma_2^2 + F_{66}\tau_{12}^2 + 2F_{12}\sigma_1\sigma_2$$

$$F_1 = \frac{1}{X_T} - \frac{1}{X_C}, \quad F_2 = \frac{1}{Y_T} - \frac{1}{Y_C}, \quad F_{11} = \frac{1}{X_T X_C}, \quad F_{22} = \frac{1}{Y_T Y_C}, \quad F_{66} = \frac{1}{S^2}, \quad F_{12} = -\frac{1}{2}\sqrt{F_{11}F_{22}}$$

Strength ratio R: failure when the applied loads are multiplied by R. With $a$ = quadratic terms, $b$ = linear terms,

$$aR^2 + bR = 1 \quad \Rightarrow \quad R = \frac{-b + \sqrt{b^2 + 4a}}{2a}$$

First-ply failure load factor = minimum R over all plies (top and bottom).

## Hashin-Rotem, 2D (Section 24.5)

$$\text{Fiber tension } (\sigma_1 \geq 0): \quad \left(\frac{\sigma_1}{X_T}\right)^2 + \left(\frac{\tau_{12}}{S_L}\right)^2 \qquad \text{Fiber compression } (\sigma_1 < 0): \quad \left(\frac{\sigma_1}{X_C}\right)^2$$

$$\text{Matrix tension } (\sigma_2 \geq 0): \quad \left(\frac{\sigma_2}{Y_T}\right)^2 + \left(\frac{\tau_{12}}{S_L}\right)^2$$

$$\text{Matrix compression } (\sigma_2 < 0): \quad \left(\frac{\sigma_2}{2S_T}\right)^2 + \left[\left(\frac{Y_C}{2S_T}\right)^2 - 1\right]\frac{\sigma_2}{Y_C} + \left(\frac{\tau_{12}}{S_L}\right)^2$$

The GUI reports the larger of the active fiber and matrix indices and its mode (FT, FC, MT, MC).

## Puck Criterion, Plane Stress

Stress exposure $f_E$ ($\geq 1$ denotes failure), homogeneous of degree 1: failure load factor $= 1/f_E$.

Inclination parameters $p_{\perp\parallel}^{(+)}$, $p_{\perp\parallel}^{(-)}$ (typical CFRP 0.35, 0.30; GFRP 0.30, 0.25).

## Fiber Failure

$$f_{E,FF} = \frac{\sigma_1}{X_T} \quad (\sigma_1 \geq 0), \qquad f_{E,FF} = \frac{-\sigma_1}{X_C} \quad (\sigma_1 < 0)$$

## Inter-Fibre Failure

$$R_{\perp\perp}^A = \frac{S}{2p_{\perp\parallel}^{(-)}} \left[ \sqrt{1 + 2p_{\perp\parallel}^{(-)} \frac{Y_C}{S}} - 1 \right], \quad p_{\perp\perp}^{(-)} = p_{\perp\parallel}^{(-)} \frac{R_{\perp\perp}^A}{S}, \quad \tau_{21c} = S\sqrt{1 + 2p_{\perp\perp}^{(-)}}$$

Mode A, $\sigma_2 \geq 0$ (transverse tension crack):

$$f_E = \sqrt{\left(\frac{\tau_{21}}{S}\right)^2 + \left(1 - p_{\perp\parallel}^{(+)} \frac{Y_T}{S}\right)^2 \left(\frac{\sigma_2}{Y_T}\right)^2} + p_{\perp\parallel}^{(+)} \frac{\sigma_2}{S}$$

Mode B, $\sigma_2 < 0$ and $|\sigma_2/\tau_{21}| \leq R_{\perp\perp}^A/\tau_{21c}$ (shear crack, compression closes it):

$$f_E = \frac{1}{S} \left[ \sqrt{\tau_{21}^2 + \left(p_{\perp\parallel}^{(-)} \sigma_2\right)^2} + p_{\perp\parallel}^{(-)} \sigma_2 \right]$$

Mode C, $\sigma_2 < 0$ and $|\sigma_2/\tau_{21}| > R_{\perp\perp}^A/\tau_{21c}$ (oblique fracture, wedge effect, delamination risk):

$$f_E = \left[ \left( \frac{\tau_{21}}{2\left(1 + p_{\perp\perp}^{(-)}\right) S} \right)^2 + \left( \frac{\sigma_2}{Y_C} \right)^2 \right] \frac{Y_C}{-\sigma_2}, \qquad \cos\theta_{fp} = \sqrt{\frac{\left(R_{\perp\perp}^A/S\right)^2 \left(\tau_{21}/\sigma_2\right)^2 + 1}{2\left(1 + p_{\perp\perp}^{(-)}\right)}}$$

The GUI reports the larger of $f_{E,FF}$ and $f_{E,IFF}$, the mode (FT, FC, A, B, C) and $\theta_{fp}$ for mode C.

Puck and Schürmann, Composites Science and Technology 62 (2002) 1633. VDI 2014 Part 3 (2006).

## LaRC05 Criterion, Plane Stress, Linear Shear

Failure index FI $\geq 1$ denotes failure. FI is not proportional to load: the load factor is found by bisection.

## Matrix Cracking (action plane, Mohr-Coulomb friction), maximum over $\theta$

$$FI_M = \left(\frac{\tau_T}{S_T - \eta_T \sigma_N}\right)^2 + \left(\frac{\tau_L}{S_L - \eta_L \sigma_N}\right)^2 + \left(\frac{\langle \sigma_N \rangle_+}{Y_T}\right)^2, \quad \sigma_N = \sigma_2 \cos^2\theta, \quad \tau_T = -\sigma_2 \sin\theta\cos\theta, \quad \tau_L = \tau_{12}\cos\theta$$

$$\eta_T = \frac{-1}{\tan 2\alpha_0}, \qquad S_T = Y_C \cos\alpha_0 \left(\sin\alpha_0 + \frac{\cos\alpha_0}{\tan 2\alpha_0}\right), \qquad \eta_L = \frac{-S_L \cos 2\alpha_0}{Y_C \cos^2\alpha_0}$$

## Fiber Failure

$$\sigma_1 \geq 0: \quad FI_{FT} = \frac{\sigma_1}{X_T}$$

$\sigma_1 < 0$: kinking ($\sigma_1 \leq -X_C/2$) or splitting, kink planes $\psi = 0$ and $90^\circ$. Misalignment (NASA TM, Eq. 101):

$$\gamma = \frac{\varphi_0 G_{12} + |\tau_{12\psi}|}{G_{12} + \sigma_1 - \sigma_{2\psi}} - \varphi_0, \qquad \varphi = \mathrm{sign}(\tau_{12\psi})\,(\varphi_0 + \gamma), \qquad \varphi_0 = \varphi_C \left(1 - \frac{X_C}{G_{12}}\right)$$

$$\varphi_C = \tan^{-1}\left[\frac{1 - \sqrt{1 - 4\left(\frac{S_L}{X_C} + \eta_L\right)\frac{S_L}{X_C}}}{2\left(\frac{S_L}{X_C} + \eta_L\right)}\right], \qquad FI_K = \left(\frac{\tau_{12m}}{S_L - \eta_L \sigma_{2m}}\right)^2 + \left(\frac{\langle \sigma_{2m} \rangle_+}{Y_T}\right)^2$$

$\sigma_{2m}$, $\tau_{12m}$: stresses rotated by $\varphi$ into the misalignment frame. $\varphi_C$ makes $FI_K = 1$ at $\sigma_1 = -X_C$.

## In-Situ Strengths (optional; Camanho et al. 2006)

$$\text{embedded}: Y_T^{is} = \max\left(\sqrt{\frac{8G_{Ic}}{\pi t \Lambda_{22}}},\ 1.12\sqrt{2}\,Y_T\right), \quad S_L^{is} = \max\left(\sqrt{\frac{8G_{12}G_{IIc}}{\pi t}},\ \sqrt{2}\,S_L\right), \quad \Lambda_{22} = 2\left(\frac{1}{E_2} - \frac{\nu_{21}^2}{E_1}\right)$$

$$\text{outer plies}: Y_T^{is} = \max\left(1.79\sqrt{\frac{G_{Ic}}{\pi t \Lambda_{22}}},\ Y_T\right), \qquad S_L^{is} = \max\left(\sqrt{\frac{4G_{12}G_{IIc}}{\pi t}},\ S_L\right)$$

Pinho et al., Composites Part A 43 (2012) 1293. Pinho et al., NASA/TM-2005-213530.

Camanho et al., Composites Part A 37 (2006) 165.

## Fatigue & Fracture Tab: Thermal Residual Stress

Lamina CTEs $\alpha_1, \alpha_2$; $\Delta T$ = service temperature − stress-free temperature (e.g. cure to room).

$$\{\alpha\}_{xy} = \left\{ \begin{array}{c} c^2\alpha_1 + s^2\alpha_2 \\ s^2\alpha_1 + c^2\alpha_2 \\ 2cs\,(\alpha_1 - \alpha_2) \end{array} \right\}$$

$$\{N^T\} = \sum_k [\bar{Q}]_k \{\alpha\}_k \,\Delta T\,(z_{k+1} - z_k), \qquad \{M^T\} = \frac{1}{2}\sum_k [\bar{Q}]_k \{\alpha\}_k \,\Delta T\,(z_{k+1}^2 - z_k^2)$$

$$\left\{ \begin{array}{c} \varepsilon_T^0 \\ \kappa_T \end{array} \right\} = \left[ \begin{array}{cc} A & B \\ B & D \end{array} \right]^{-1} \left\{ \begin{array}{c} N^T \\ M^T \end{array} \right\}, \qquad \{\sigma\}_{xy}^r = [\bar{Q}]_k \left(\{\varepsilon_T^0\} + z\{\kappa_T\} - \{\alpha\}_k \Delta T\right)$$

The residual stress is a static stress added to the mechanical stress: $\{\sigma\} = \lambda\{\sigma\}^m + \{\sigma\}^r$.

Tsai-Wu load factor with residual stress, $q(\,,\,)$ = quadratic part of Tsai-Wu:

$$a\lambda^2 + b\lambda + c = 0, \quad a = q(\sigma^m, \sigma^m), \quad b = 2q(\sigma^m, \sigma^r) + F_1\sigma_1^m + F_2\sigma_2^m, \quad c = q(\sigma^r, \sigma^r) + F_1\sigma_1^r + F_2\sigma_2^r - 1$$

## Progressive Ply Failure (ply discount)

1. Lowest load factor $\lambda$: Tsai-Wu (or Puck) for intact plies, fiber maximum stress for matrix-failed plies.

2. The Hashin-Rotem (or Puck) mode sets the discount. Matrix: $E_2, G_{12}, \nu_{12} \times 10^{-4}$. Fiber: also $E_1 \times 10^{-4}$.

Puck or LaRC05 with residual stress: $f(\lambda\sigma^m + \sigma^r) = 1$ is solved for $\lambda$ by bisection.

3. Recompute ABD and the residual stress. Repeat, with $\lambda$ never decreasing.

Ultimate = first fiber failure. Collapse when $\{N, M\}^T [ABD]^{-1} \{N, M\}$ is 100 times the intact value.

## Open-Hole Strength, Whitney-Nuismer

Orthotropic infinite plate, hole radius $R = D/2$, loading in x (Lekhnitskii):

$$K_T^\infty = 1 + \sqrt{2\left(\sqrt{E_x/E_y} - \nu_{xy}\right) + \frac{E_x}{G_{xy}}}$$

Stress ahead of the hole, approximate (Konish and Whitney), $y$ from the hole center:

$$\frac{\sigma_x(0,y)}{\sigma_\infty} = \frac{1}{2}\left[2 + \xi^2 + 3\xi^4 - (K_T^\infty - 3)\left(5\xi^6 - 7\xi^8\right)\right], \qquad \xi = \frac{R}{y}$$

Point stress: $\sigma_x = \sigma_0$ at distance $d_0$ from the hole edge

$$\frac{\sigma_N}{\sigma_0} = \frac{2}{2 + \xi_1^2 + 3\xi_1^4 - (K_T^\infty - 3)(5\xi_1^6 - 7\xi_1^8)}, \qquad \xi_1 = \frac{R}{R + d_0}$$

Average stress: the average of $\sigma_x$ over distance $a_0$ from the hole edge equals $\sigma_0$

$$\frac{\sigma_N}{\sigma_0} = \frac{2(1 - \xi_2)}{2 - \xi_2^2 - \xi_2^4 + (K_T^\infty - 3)(\xi_2^6 - \xi_2^8)}, \qquad \xi_2 = \frac{R}{R + a_0}$$

Unnotched strength $\sigma_0$ = first fiber failure of the progressive analysis under $N_x$ alone, divided by $h$.

If no ply fails in a fiber mode before the laminate collapses (no fibers near x), $\sigma_0$ = collapse load / $h$.

Finite width $W$ (isotropic correction): $\sigma_N = \sigma_N^\infty/\mathrm{FWC}$

$$\mathrm{FWC} = \frac{2 + (1 - D/W)^3}{3\,(1 - D/W)}$$

Whitney and Nuismer, J. Composite Materials 8 (1974) 253. Tests: ASTM D5766 (OHT), ASTM D6484 (OHC).

## Free-Edge Delamination Onset, O'Brien

$$G = \frac{\varepsilon_x^2\, h}{2}\,(E_x - E^*)\,, \qquad E^* = \frac{1}{h}\sum_i E_i\, t_i, \qquad E_i = \frac{1}{t_i\,(abd)^i_{11}}$$

$E_i$ uses the ABD matrix of sublaminate $i$ about its own mid-plane, so an unsymmetric sublaminate is free to bend.

$$\varepsilon_{onset} = \sqrt{\frac{2\,G_c}{h\,(E_x - E^*)}}, \qquad \frac{G_{max}}{G_c} = 1 - b_G \log_{10} N$$

Mechanical $\varepsilon_x$. O'Brien, ASTM STP 775 (1982) 140. Toughness: ASTM D5528 (Mode I), D7905 (Mode II).

## Damage-Tolerance Strain Check

Principal strains at the surfaces, $\varepsilon^0 \pm (h/2)\,\kappa$, mechanical loads only, compared with

$$\varepsilon_{T,allow} = k_d\,\varepsilon_{OHT}, \qquad \varepsilon_{C,allow} = k_d\,\min(\varepsilon_{OHC},\ \varepsilon_{CAI}), \qquad MS = \frac{\varepsilon_{allow}}{|\varepsilon|} - 1$$

See CMH-17-3G Vol. 3 and FAA AC 20-107B. CAI tests: ASTM D7136, D7137.

## Ply Fatigue, Hashin-Rotem

Each stress component at the maximum and minimum load, residual stress included:

$$\sigma_a = \frac{|\sigma_{max} - \sigma_{min}|}{2}, \qquad \sigma_m = \frac{\sigma_{max} + \sigma_{min}}{2}$$

Linear constant-life diagram through the R = −1 curve and the static strengths ($Y_T, Y_C$ for $\sigma_2$; $S$ and $|\tau_m|$ for $\tau_{12}$):

$$\sigma_{a,eq} = \frac{\sigma_a}{1 - \sigma_m / X_T} \quad (\sigma_m \geq 0), \qquad \sigma_{a,eq} = \frac{\sigma_a}{1 + \sigma_m / X_C} \quad (\sigma_m < 0)$$

S-N curves at R = −1, $X_{ref} = \min(X_T, X_C)$, $Y_{ref} = \min(Y_T, Y_C)$, slopes $b_1, b_2, b_6$ per decade:

$$\text{Fiber :} \quad \frac{\sigma_{1a,eq}}{X_{ref}} = 1 - b_1 \log_{10} N$$

$$\text{Matrix :} \quad \left( \frac{\sigma_{2a,eq}}{Y_{ref}\,(1 - b_2 \log_{10} N)} \right)^2 + \left( \frac{\tau_{12a,eq}}{S\,(1 - b_6 \log_{10} N)} \right)^2 = 1$$

Hashin and Rotem, J. Composite Materials 7 (1973) 448.

## Load Histories, Miner's Rule

$$D = \sum_i \frac{n_i}{N_i}, \qquad D \geq 1 \text{ denotes failure}$$

Constant amplitude: max = applied loads, min = R × applied loads.

Block spectrum: max = factor × applied loads, min = R × max.

Narrowband random: applied loads = 1-sigma, zero mean, Rayleigh peaks $\lambda$ (peak / 1-sigma), N cycles:

$$D = N \int_0^8 \frac{\lambda\, e^{-\lambda^2/2}}{N_f(\lambda)}\, d\lambda$$

Fiber and matrix damage are summed separately. Miner's rule is often non-conservative for composites.

## Units

### English

Moduli $E_1, E_2, G_{12},\ E_f, E_m, G_f, G_m$ : psi

Strengths and stresses : psi

Ply thickness, z : in

Force resultants $N_x, N_y, N_{xy}$ : lbf/in

Moment resultants $M_x, M_y, M_{xy}$ : in-lbf/in

A : lbf/in, B : lbf, D : lbf-in, curvature : 1/in

CTE : microstrain/F, $\Delta T$ : F, $G_c$, $G_{Ic}$, $G_{IIc}$ : in-lbf/in$^2$, hole D, W, $d_0$, $a_0$ : in

### metric

Moduli : GPa

Strengths and stresses : MPa

Ply thickness, z : mm

Force resultants : N/m

Moment resultants : N-m/m

A : N/m, B : N, D : N-m, curvature : 1/m

CTE : microstrain/C, $\Delta T$ : C, $G_c$, $G_{Ic}$, $G_{IIc}$ : J/m$^2$, hole D, W, $d_0$, $a_0$ : mm

Mid-plane strains and strain allowables are in microstrain.

Poisson ratios, $V_f$, Puck $p$, S-N slopes and $k_d$ are dimensionless.

Default values are typical carbon/epoxy example values. Replace them with test data or design allowables.