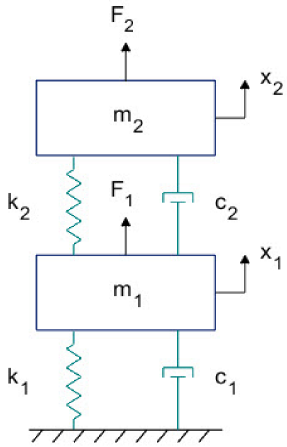
# Two-DOF System with Dashpots: Equations of Motion

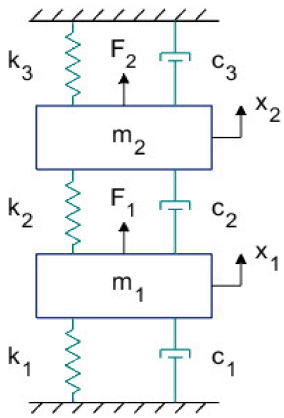


$x_1$, $x_2$ displacements
$F_1$, $F_2$ applied forces
m mass
c damping coefficient
k stiffness

System 1 is System 2 with $k_3 = c_3 = 0$.




**System 1**




**System 2**

## Newton's Law for Each Mass (System 2)

$$m_1\ddot{x}_1 + c_1\dot{x}_1 + c_2(\dot{x}_1 - \dot{x}_2) + k_1x_1 + k_2(x_1 - x_2) = F_1$$

$$m_2\ddot{x}_2 + c_2(\dot{x}_2 - \dot{x}_1) + c_3\dot{x}_2 + k_2(x_2 - x_1) + k_3x_2 = F_2$$

## Matrix Form

$$M\ddot{x} + C\dot{x} + Kx = F\ , \qquad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad F = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \quad C = \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 + c_3 \end{bmatrix} \quad K = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}$$

System 1: set $c_3 = k_3 = 0$, so that $C_{22} = c_2$ and $K_{22} = k_2$.

## Why a State-Space Formulation?

The normal modes of K and M uncouple the damped equations only if C is proportional, for example C = $\alpha$ M + $\beta$ K. Dashpot values are arbitrary, so in general the damping is non-proportional and the modes are complex. Rewriting the second-order 2x2 system as a first-order 4x4 system gives an eigenvalue problem that is valid for any C (page 2).

## Units

| Units | Mass | Damping | Stiffness |
|---|---|---|---|
| English | lbm, divided by 386 $\rightarrow$ lbf sec$^2$/in | lbf sec/in | lbf/in |
| metric | kg | N sec/m | N/m |

g = 386 in/sec$^2$ = 9.81 m/sec$^2$. Accelerance FRFs are reported in G per unit force.

# State-Space Partitioning and the Generalized Eigenvalue Problem



## Step 1. Add an Identity Equation

Keep the equation of motion and add a trivial equation so that the system has the same size in x and in its derivative:

$$C\dot{x} + M\ddot{x} + Kx = F \qquad\qquad M\dot{x} - M\dot{x} = 0$$

## Step 2. Partition into 4x4 Matrices

$$\begin{bmatrix} C & M \\ M & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} + \begin{bmatrix} K & 0 \\ 0 & -M \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} F \\ 0 \end{bmatrix}$$

$$A\dot{z} + Bz = Q\ , \qquad z = \begin{bmatrix} x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}, \quad Q = \begin{bmatrix} F \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} c_1 + c_2 & -c_2 & m_1 & 0 \\ -c_2 & c_2 + c_3 & 0 & m_2 \\ m_1 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 & 0 \\ -k_2 & k_2 + k_3 & 0 & 0 \\ 0 & 0 & -m_1 & 0 \\ 0 & 0 & 0 & -m_2 \end{bmatrix}$$

A and B are real and symmetric. The app shows them on the View Modal Results > 4x4 Matrices tab.

## Step 3. Generalized Eigenvalue Problem (free vibration, Q = 0)

$$z = \psi\, e^{\lambda t} \quad \Rightarrow \quad (\lambda A + B)\,\psi = 0\ , \qquad \det(\lambda A + B) = 0$$

Four eigenvalues $\lambda_r$, in complex-conjugate pairs for an underdamped system.

Each eigenvector is partitioned as displacement over velocity:

$$\psi_r = \begin{bmatrix} \phi_r \\ \lambda_r \phi_r \end{bmatrix}\ , \qquad \psi_r^T A\, \psi_r = 1 \quad \text{(normalization)}$$

## Solution Used in the App

$$\dot{z} = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix} z$$

has the same eigenvalues and eigenvectors.
The app solves it with eig, then scales each $\psi_r$.

$$\lambda_r = -\zeta_r\omega_r \pm i\,\omega_r\sqrt{1-\zeta_r^2}\ , \quad \omega_r = |\lambda_r|\ , \quad f_r = \frac{\omega_r}{2\pi}\ , \quad \zeta_r = \frac{-\mathrm{Re}(\lambda_r)}{|\lambda_r|}$$

Foss, K.A., Coordinates Which Uncouple the Equations of Motion of Damped Linear Dynamic Systems, Journal of Applied Mechanics, Vol. 25, 1958.
Hurty, W.C. and Rubinstein, M.F., Dynamics of Structures, Prentice-Hall, 1964.