Vibrationdata Fatigue & Fracture: Equations and Units

The calculations run in English units. The Units selector converts inputs on entry and results on display.

Units

Quantity	English	Metric	Conversion
Stress S, strength	ksi	MPa	1 ksi = 6.894757 MPa
Crack size a, c, width, thickness	in	mm	1 in = 25.4 mm
Stress intensity K, dK	ksi sqrt(in)	MPa sqrt(m)	1 ksi sqrt(in) = 1.098843 MPa sqrt(m)
Crack growth rate da/dN	in/cycle	mm/cycle	1 in/cycle = 25.4 mm/cycle
Stress PSD	ksi^2/Hz	MPa^2/Hz	1 ksi^2/Hz = 47.538 MPa^2/Hz
Base acceleration PSD (FDS, ERS)	G^2/Hz	G^2/Hz	none
Contact force, load per length	lbf, lbf/in	N, N/mm	1 lbf = 4.448222 N
EHL speed, roughness, film; alpha	in/s, uin; 1/ksi	m/s, um; 1/GPa	1/ksi = 145.04 1/GPa; viscosity cP

Growth-law constant C (da/dN = C dK^n form)

$$C_{SI} = C_{English} \frac{25.4}{1.098843^{\,n}}$$

The converted C depends on n, so set n before typing C in metric units.

Notation

S far-field stress, R = Smin/Smax, a crack depth or length, c surface half-length, W width, t thickness, D hole diameter

N cycles, Sa stress amplitude, Sm mean stress, b fatigue exponent, Q amplification factor, T duration

Material constants are representative values: verify against test data or an approved database before design use.

Crack Growth Equations

Forman-Newman-de Koning (FNK)

$$\frac{da}{dN} = C\left[\frac{1-f}{1-R}\,\Delta K\right]^{n}\frac{(1-\Delta K_{th}/\Delta K)^{p}}{(1-K_{max}/K_{c})^{q}}$$

Newman crack-opening function

$$f = \max\left(R,\ A_0 + A_1 R + A_2 R^2 + A_3 R^3\right),\ R \geq 0 \qquad f = A_0 + A_1 R,\ -2 \leq R < 0$$

$$A_0 = \left(0.825 - 0.34\alpha + 0.05\alpha^2\right)\left[\cos\left(\frac{\pi S_{max}}{2\sigma_0}\right)\right]^{1/\alpha}, \quad A_1 = (0.415 - 0.071\alpha)\,\frac{S_{max}}{\sigma_0}$$

$$A_3 = 2A_0 + A_1 - 1, \qquad A_2 = 1 - A_0 - A_1 - A_3$$

Threshold and toughness

$$\Delta K_{th} = \Delta K_0\sqrt{\frac{a}{a+a_0}}\Bigg/\left[\frac{1-f}{(1-A_0)\,(1-R)}\right]^{1+C_{th}R}$$

$$K_c = K_{Ic}\left[1 + B_k\, e^{-(A_k t/t_0)^2}\right], \qquad t_0 = 2.5\,(K_{Ic}/\sigma_{ys})^2$$

Paris and Walker

$$\frac{da}{dN} = C_p\,\Delta K^{n_p} \qquad \frac{da}{dN} = C_p\left[\frac{\Delta K}{(1-R)^{1-\gamma}}\right]^{n_p}, \qquad C_p = C\,(1-A_0)^n$$

For R < 0, Paris and Walker use Kmax in place of dK (positive part of the cycle).

Load Interaction and Tabulated Growth Rates

Generalized Willenborg

$$Z = \frac{1}{\pi\,\alpha_c}\left(\frac{K_{max}}{\sigma_{ys}}\right)^2, \qquad K_{req} = \sigma_{ys}\sqrt{\pi\,\alpha_c\,(a_{OL} + Z_{OL} - a)}$$

$$\phi = \frac{1 - K_{max,th}/K_{max}}{R_{so} - 1}, \qquad K_R = \phi\,(K_{req} - K_{max})\,, \qquad K_{max,eff} = K_{max} - K_R$$

Kmin,eff = max(Kmin - KR, 0) when Kmin > 0. Growth arrests when Kreq/Kmax >= Rso.

Constant closure

$$S_{op} = C_f\,S_{max,block}, \qquad \Delta K_{eff} = K_{max} - \max\left(S_{op}\,k(a),\,K_{min}\right)$$

$$C_f = f\,(R_{spec})\,, \qquad R_{spec} = \frac{\min S_{min}}{\max S_{max}} \quad (\text{automatic})$$

dKeff replaces the closure-corrected range in FNK; cycles with Kmax below Kop do not grow. Cf can be calibrated to a test life.

Tabulated da/dN

$$\log\frac{da}{dN} = (1 - w)\log r_i\,(\Delta K) + w\,\log r_{i+1}\,(\Delta K)\,, \qquad w = \frac{R - R_i}{R_{i+1} - R_i}$$

Each curve is interpolated log-log in dK; the rate is zero below a curve's first dK.

Strip-Yield Crack Closure (Loading tab, Retardation)

Newman (1981): bar elements in the plastic zone (intact) and in the wake (broken) of a crack of half-length c, infinite plate.

$$\rho = c\left[\sec\frac{\pi S_{max}}{2\alpha\sigma_0} - 1\right], \qquad d = c + \rho, \qquad \sigma_0 = \frac{\sigma_{ys} + \sigma_u}{2}$$

$$V(x) = \frac{2}{E}\left[S\sqrt{d^2 - x^2} - \frac{1}{\pi}\sum_j \sigma_j\, g\left(x, b_{1j}, b_{2j}\right)\right], \qquad g = G\left(x, b_2\right) - G\left(x, b_1\right)$$

$$G(x,b) = (b-x)\cosh^{-1}\frac{d^2 - bx}{d\,|b-x|} + (b+x)\cosh^{-1}\frac{d^2 + bx}{d\,|b+x|} + 2\sqrt{d^2 - x^2}\,\sin^{-1}\frac{b}{d}$$

At Smax and Smin: V(xi) = Li (residual stretch) for each element; plastic zone -s0 <= sj <= alpha s0, wake -s0 <= sj <= 0 (open when V > L). Yielded elements take the new length. Solved by an active set on the bounds.

$$S_o = S_{min} - \frac{2}{\pi}\sum_{wake}\sigma_j\left[\sin^{-1}\frac{b_{2j}}{c} - \sin^{-1}\frac{b_{1j}}{c}\right], \qquad \Delta K_{eff} = K_{max} - \max\left(K_o,\, K_{min}\right)$$

dKeff replaces the closure-corrected range in FNK. The model is updated with the largest Smax and smallest Smin since the last update after 1% of the plastic zone of growth, and at once on a new overload or underload. 20 plastic-zone elements, 80 wake.

Checks

g matches the integral of the point-load solution (1e-11); the crack-tip opening matches Dugdale; the reversed zone matches c[sec(pi dS / (2(1 + alpha) s0)) - 1] within one element; constant-amplitude So/Smax is within 6% of Newman's closure equations (alpha = 1 to 3, R = -1 to 0.5), and the constant-amplitude life within 10% of FNK.

Limits

Infinite-plate opening stress with the geometry's K; fixed alpha (no constraint-loss transition); surface and corner cracks use the surface length c for So, applied to both tips; a part-through to through transition starts a new wake.

Stress Intensity Factors

$$K = \beta\, S\sqrt{\pi a}$$

Center through crack (Tada)

$$\beta = \left(1 - 0.025\lambda^2 + 0.06\lambda^4\right)\sqrt{\sec\frac{\pi\lambda}{2}}, \qquad \lambda = \frac{2a}{W}$$

Edge through crack (Tada)

$$\beta = \sqrt{\frac{\tan x}{x}}\;\frac{0.752 + 2.02\lambda + 0.37\left(1 - \sin x\right)^3}{\cos x}, \qquad x = \frac{\pi\lambda}{2},\;\; \lambda = \frac{a}{W}$$

Surface and corner cracks (Newman-Raju)

$$K = \left[M_1 + M_2\left(\frac{a}{t}\right)^2 + M_3\left(\frac{a}{t}\right)^4\right] g\, f_\phi\, f_w\, S\sqrt{\frac{\pi a}{Q}}, \qquad Q = 1 + 1.464\left(\frac{a}{c}\right)^{1.65}$$

Weight functions (stress gradient p(x) = stress / S)

$$K_{CC} = 2\sqrt{\frac{a}{\pi}}\int_0^a \frac{p(x)}{\sqrt{a^2 - x^2}}\,dx \times \beta_{Tada}\left(a/W\right)$$

$$m_A = \frac{2}{\sqrt{2\pi\left(a - x\right)}}\left[1 + M_1 s^{1/2} + 3s + M_3 s^{3/2}\right], \quad s = 1 - \frac{x}{a}$$

$$m_C = \frac{2}{\sqrt{\pi x}}\left[1 + N_1 v + N_2 v^2 + N_3 v^3\right], \quad v = \sqrt{x/a},\;\; 1 + N_1 + N_2 + N_3 = 0$$

M1, M3 and N1, N3 are fitted at each crack size to the uniform and linear (bending) reference solutions.

Residual Strength and Inspection

LEFM with net-section yield

$$S_f = \min\left(\frac{K_c}{\beta\sqrt{\pi a}},\ \frac{\sigma_{flow}}{A_g/A_n}\right), \qquad \sigma_{flow} = \frac{\sigma_{ys}+\sigma_u}{2}$$

Failure assessment diagram (R6 / BS 7910 Option 1)

$$K_r = \frac{K}{K_c}, \qquad L_r = \frac{\sigma_{ref}}{\sigma_{ys}}, \qquad L_{r,max} = \frac{\sigma_{ys}+\sigma_u}{2\,\sigma_{ys}}$$

$$f\left(L_r\right) = \left(1+\frac{L_r^2}{2}\right)^{-1/2}\left[0.3+0.7\,e^{-0.6L_r^6}\right]$$

The reference stress is the net-section stress (a simplification for surface cracks). Check against the current standard.

R-curve tangency (through cracks)

$$S_c = \max_{\Delta a}\ \frac{K_R\left(\Delta a\right)}{\beta\left(a_0+\Delta a\right)\sqrt{\pi\left(a_0+\Delta a\right)}} \quad \text{(first maximum)}$$

Inspection interval

$$\Delta N_{insp} = \frac{N\left(a_{crit}\right) - N\left(a_{det}\right)}{SF}$$

Stress-Life Fatigue

S-N curve (stress amplitude)

$$S_a = S_f\,(2N)^b \qquad \text{or} \qquad N = N_1\left(\frac{S_a}{S_{a1}}\right)^{-m_1}$$

$$N = N_k\left(\frac{S_a}{S_k}\right)^{-m_2}, \quad N > N_k \qquad (m_2 = \infty:\ \text{endurance limit})$$

Strength modification (Marin)

$$k_f = \frac{k_a\,k_b\,k_c\,k_d\,k_e}{K_f}, \qquad k_a = a\,S_u^{b_s}, \qquad S_{a,allow} = k_f\,S_a\,(N)$$

ka, kb, ke are textbook estimates for steels (ksi, inch); check against your source.

Mean stress corrections (equivalent fully reversed amplitude)

$$\text{Goodman}: \frac{S_a}{1 - S_m/S_u} \quad \text{Gerber}: \frac{S_a}{1 - (S_m/S_u)^2} \quad \text{Morrow}: \frac{S_a}{1 - S_m/S_f}$$

$$\text{SWT}: \sqrt{S_{max}\,S_a} \qquad \text{Walker}: S_{max}^{1-\gamma}\,S_a^{\gamma}$$

Miner damage

$$D = \sum_i \frac{n_i}{N_i}, \qquad \text{Life} = \frac{1}{D}\ \text{blocks}$$

Strain-Life and Weld Fatigue

Strain-life (Coffin-Manson-Basquin) with Neuber

$$\varepsilon_a = \frac{\sigma'_f}{E}(2N)^b + \varepsilon'_f(2N)^c$$

$$\varepsilon = \frac{\sigma}{E} + \left(\frac{\sigma}{K'}\right)^{1/n'}, \qquad \Delta\sigma\,\Delta\varepsilon = \frac{(K_t\,\Delta S)^2}{E}$$

$$\mathrm{SWT}: \sigma_{max}\,\varepsilon_a = \frac{\left(\sigma'_f\right)^2}{E}(2N)^{2b} + \sigma'_f\,\varepsilon'_f(2N)^{b+c}$$

Uniform Material Law estimates

Steel: sf' = 1.5 Su, b = -0.087, ef' = 0.59 psi, c = -0.58. Al, Ti: sf' = 1.67 Su, b = -0.095, ef' = 0.35, c = -0.69.

Weld S-N (stress range, FAT = range at 2 million cycles)

$$N = 2\times10^6\left(\frac{FAT}{\Delta\sigma}\right)^{m_1}, \qquad N = N_k\left(\frac{\Delta\sigma_k}{\Delta\sigma}\right)^{m_2}, \;\; \Delta\sigma < \Delta\sigma_k$$

$$FAT_t = FAT\left(\frac{t_{ref}}{t}\right)^{n_t}, \;\; t > t_{ref}$$

Random Vibration Fatigue: Spectral Methods

Spectral moments, rates and damage

$$m_j = \int_0^\infty f^j\, G(f)\, df, \quad \nu_0 = \sqrt{\frac{m_2}{m_0}}, \quad \nu_p = \sqrt{\frac{m_4}{m_2}}, \quad \alpha_1 = \frac{m_1}{\sqrt{m_0 m_2}}, \quad \alpha_2 = \frac{m_2}{\sqrt{m_0 m_4}}$$

$$D = \nu\, T \int_0^\infty \frac{p(S_a)}{N(S_a)}\, dS_a, \qquad \sigma = \sqrt{m_0}, \quad z = S_a/\sigma$$

$$\text{Narrowband}: \; p = \frac{z}{\sigma}\, e^{-z^2/2}, \;\; \nu = \nu_0$$

$$\text{Dirlik}: \; p = \frac{1}{\sigma}\left[\frac{D_1}{Q} e^{-z/Q} + \frac{D_2 z}{R^2} e^{-z^2/2R^2} + D_3 z\, e^{-z^2/2}\right], \;\; \nu = \nu_p$$

$$\text{Zhao}-\text{Baker}: \; p = \frac{1}{\sigma}\left[w\, \alpha\beta z^{\beta-1} e^{-\alpha z^\beta} + (1-w)\, z\, e^{-z^2/2}\right], \;\; \alpha = 8 - 7\alpha_2$$

$$\text{Tovo}-\text{Benasciutti}: \; D = \left[b + (1-b)\, \alpha_2^{k-1}\right] D_{NB}$$

$$\text{Lalanne (Rice peaks)}, \;\; \nu = \nu_p; \qquad \text{Steinberg}: 68.3\% \text{ at } 1\sigma,\; 27.1\% \text{ at } 2\sigma,\; 4.33\% \text{ at } 3\sigma$$

Non-Gaussian (Hermite) correction

$$x = \kappa\left[u + h_4\left(u^3 - 3u\right)\right], \qquad \kappa = \left(1 + 6h_4^2\right)^{-1/2}$$

h4 is solved so the model kurtosis equals the input kurtosis; amplitudes become sigma x(Sa/sigma).

Fatigue Damage Spectrum, ERS and Test Tailoring

SDOF absolute acceleration response to base input

$$H(f) = \frac{\omega_n^2 + i\,2\zeta\omega\omega_n}{\omega_n^2 - \omega^2 + i\,2\zeta\omega\omega_n}, \qquad \zeta = \frac{1}{2Q}, \qquad G_{resp} = |H|^2\, G_{base}$$

Fatigue damage spectrum (pseudo-damage, A = 1)

$$FDS(f_n) = \nu\, T \int_0^\infty p(S_a)\, S_a^b\, dS_a \qquad \left[\mathrm{G}^b\right]$$

Extreme response spectrum

$$ERS(f_n) = \sigma_{resp}\sqrt{2\ln(\nu_0 T)}, \qquad \sigma_{resp} \approx \sqrt{\frac{\pi}{2} f_n\, Q\, G(f_n)}$$

Time compression at equal damage

$$G_{test} = G_{field}\left(\frac{T_{field}}{T_{test}}\right)^{2/b}$$

Test PSD from an FDS envelope

$$G(f_n) = \frac{2\sigma^2}{\pi f_n Q}, \quad \sigma = \frac{1}{\sqrt{2}}\left[\frac{FDS}{f_n T\,\Gamma(1 + b/2)}\right]^{1/b}, \quad G \leftarrow G\left(\frac{FDS_{target}}{FDS}\right)^{2/b}$$

Unit-Load Superposition, Multiaxial Fatigue and Biaxiality

Superposition

$$\{\sigma_x, \sigma_y, \tau_{xy}\}(t) = \sum_j \{\sigma_x, \sigma_y, \tau_{xy}\}_j^{unit} \; L_j(t)$$

Critical plane (plane normal at angle theta from x)

$$\sigma_\theta = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\tau_\theta = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta, \qquad S_{a,eq} = \sqrt{3}\,\tau_a \;\; (\text{shear})$$

Damage is rainflow counted on every plane; the critical plane has the largest damage.

Principal stresses and scalar measures

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}, \qquad \sigma_{vm} = \sqrt{\sigma_x^2 - \sigma_x\sigma_y + \sigma_y^2 + 3\tau_{xy}^2}$$

Biaxiality

$$a = \frac{\sigma_2}{\sigma_1}, \quad |\sigma_1| \geq |\sigma_2| \qquad a = 0 \text{ uniaxial}, \;\; -1 \text{ shear}, \;\; +1 \text{ equibiaxial}$$

Proportional loading: principal-angle spread < 5 deg and std(a) < 0.1. Otherwise use a critical-plane measure.

Multiaxial Fatigue: Stress History (Multiaxial tab)

Stress on a plane (3D, normal n, shear direction t in the plane)

$$\sigma_n = \mathbf{n}^T [\sigma] \mathbf{n}, \qquad \tau = \mathbf{t}^T [\sigma] \mathbf{n}, \qquad \mathbf{n} = \{\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta\}$$

Every plane is searched (theta 0-180 deg from z, phi 0-180 deg from x, shear direction psi), then refined at a quarter step.

Findley critical plane

$$f = \tau_a + k\,\sigma_{n,max}, \qquad S_{a,eq} = \frac{2f}{k + \sqrt{1+k^2}}$$

$$\frac{t_{-1}}{f_{-1}} = \frac{k + \sqrt{1+k^2}}{2\sqrt{1+k^2}} \qquad (k = 0:\ S_{a,eq} = 2\tau_a,\ \text{Tresca})$$

tau_a from rainflow of tau on each plane and direction; sigma_n,max is the largest normal stress on the plane

during each counted cycle (ver 1.4; over the whole history in ver 1.3).

Seq is exact for uniaxial loading; damage = sum n / N(Seq) on the S-N tab curve.

Normal-stress criterion: rainflow of sigma_n on each plane with the S-N tab mean stress correction.

Crossland and Sines (endurance, constant amplitude)

$$\sqrt{J_{2,a}} + \alpha\,\sigma_h \le t_{-1}, \qquad \sigma_h = \frac{\sigma_x + \sigma_y + \sigma_z}{3}$$

$$\text{Crossland}:\ \sigma_h = \sigma_{h,max},\ \ \alpha = \frac{3t_{-1}}{f_{-1}} - \sqrt{3} \qquad \text{Sines}:\ \sigma_h = \sigma_{h,mean},\ \ \alpha = \frac{6t_{-1}}{f_0} - \sqrt{3}$$

sqrt(J2,a) = radius of the smallest sphere enclosing the deviatoric stress path; f0 = 2 f-1 Su / (f-1 + Su) by Goodman if not given.

Multiaxial Fatigue: Stress PSD (Multiaxial tab)

Stress cross-PSD from transfer functions (single input)

$$\mathbf{G}_\sigma(f) = \mathbf{H}^*(f)\,\mathbf{H}^T(f)\,G_{in}(f), \qquad \mathbf{H} = \{H_{\sigma x}, H_{\sigma y}, H_{\sigma z}, H_{\tau xy}, H_{\tau yz}, H_{\tau xz}\}$$

Equivalent von Mises stress PSD (Preumont and Piefort, 1994)

$$G_{vm}(f) = \sum_{i,j} Q_{ij}\,\mathrm{Re}\,G_{ij}(f)$$

$$\mathbf{Q} = \begin{bmatrix} 1 & -0.5 & -0.5 & 0 & 0 & 0 \\ -0.5 & 1 & -0.5 & 0 & 0 & 0 \\ -0.5 & -0.5 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix} \quad \text{for} \quad [\sigma_x\ \sigma_y\ \sigma_z\ \tau_{xy}\ \tau_{yz}\ \tau_{xz}]$$

Critical plane in the frequency domain

$$G_n(f) = \mathbf{c}_n\,\mathbf{G}_\sigma\,\mathbf{c}_n^T, \qquad G_{2\tau}(f) = 4\,\mathbf{c}_\tau\,\mathbf{G}_\sigma\,\mathbf{c}_\tau^T$$

c: coefficients of n^T sigma n (normal) or t^T sigma n (shear); the plane with the largest damage is critical.

Damage of the equivalent PSD uses the selected spectral method (Dirlik etc.) and the S-N tab curve.

Von Mises is positive-only, so it suits zero-mean loading; the critical-plane methods keep the sign of the stress on the plane.

Contact Fatigue: Hertz Contact Stresses (Contact tab)

Contact size and peak pressure

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}, \qquad \frac{1}{E^*} = \frac{1-\nu_1^2}{E_1} + \frac{1-\nu_2^2}{E_2}$$

$$\text{Line}: \; a = \sqrt{\frac{4wR}{\pi E^*}}, \;\; p_0 = \frac{2w}{\pi a} \qquad \text{Point}: \; a = \left(\frac{3PR}{4E^*}\right)^{1/3}, \;\; p_0 = \frac{3P}{2\pi a^2}$$

Spur gears at the pitch point: R1 = (d1/2) sin(phi), R2 = (d2/2) sin(phi), w = Wt / (F cos(phi)). Flat: R = 0; concave: R < 0.

Line contact with traction q = mu p (McEwen; Johnson, Contact Mechanics, 4.2 and 7.1)

$$m^2,\, n^2 = \frac{1}{2}\left[\sqrt{\left(a^2 - x^2 + z^2\right)^2 + 4x^2z^2} \pm \left(a^2 - x^2 + z^2\right)\right], \quad \operatorname{sign} n = \operatorname{sign} x$$

$$\sigma_x = -\frac{p_0}{a}\left[m\left(1 + \frac{z^2+n^2}{m^2+n^2}\right) - 2z\right] + \frac{q_0}{a}\left[n\left(2 + \frac{m^2-z^2}{m^2+n^2}\right) - 2x\right]$$

$$\sigma_z = -\frac{p_0}{a}\, m\left(1 - \frac{z^2+n^2}{m^2+n^2}\right) - \frac{q_0}{a}\, n\, \frac{m^2-z^2}{m^2+n^2}$$

$$\tau_{xz} = -\frac{p_0}{a}\, n\, \frac{m^2-z^2}{m^2+n^2} - \frac{q_0}{a}\left[m\left(1 + \frac{z^2+n^2}{m^2+n^2}\right) - 2z\right], \qquad \sigma_y = \nu\left(\sigma_x + \sigma_z\right)$$

q0 = mu p0 acts on body 1 in +x; plane strain. Pure rolling: tau_max = 0.300 p0 at z = 0.786 a; orthogonal shear 0.25 p0 at 0.5 a.

Contact Fatigue: Point Contact and Rolling Histories (Contact tab)

Point (circular) contact, frictionless (Huber; Johnson eq. 3.45)

$$u = \frac{1}{2}\left[r^2 + z^2 - a^2 + \sqrt{\left(r^2 + z^2 - a^2\right)^2 + 4a^2 z^2}\right], \qquad \zeta = \frac{z}{\sqrt{u}}, \qquad T = \frac{\sqrt{u}}{a}\tan^{-1}\frac{a}{\sqrt{u}}$$

$$\frac{\sigma_r}{p_0} = \frac{1-2\nu}{3}\frac{a^2}{r^2}\left(1-\zeta^3\right) + \zeta^3\frac{a^2 u}{u^2 + a^2 z^2} + \zeta\left[\frac{(1-\nu)\,u}{a^2+u} + (1+\nu)\,T - 2\right]$$

$$\frac{\sigma_\theta}{p_0} = -\frac{1-2\nu}{3}\frac{a^2}{r^2}\left(1-\zeta^3\right) - \zeta\left[2\nu + \frac{(1-\nu)\,u}{a^2+u} - (1+\nu)\,T\right]$$

$$\frac{\sigma_z}{p_0} = -\zeta^3\frac{a^2 u}{u^2 + a^2 z^2}, \qquad \frac{\tau_{rz}}{p_0} = -\frac{r z^2}{u^2 + a^2 z^2}\frac{a^2\sqrt{u}}{a^2+u}$$

On the axis tau_max = 0.31 p0 at z = 0.48 a (nu = 0.3). Checked against the axis closed form and equilibrium (div sigma = 0).

Stress history of a material point

As the contact rolls past, a point at depth z sees the field along x: one load cycle per pass, x = -4a to +4a.

$$[\sigma](t) = [\sigma]^{Hertz}(x(t), z) + [\sigma]^{res}, \qquad [\sigma]^{res} = \mathrm{diag}\left(\sigma_x^{res},\, \sigma_y^{res},\, 0\right)$$

Per depth: largest Tresca shear, orthogonal shear amplitude (half range of tau_xz), largest von Mises stress.

Point contact: the track centerline y = 0, with r, theta stresses rotated to x, y, z.

Contact Fatigue: Dang Van Criterion (Contact tab)

Dang Van criterion

$$\hat{\mathbf{s}}(t) = \mathbf{s}(t) - \mathbf{s}^*, \qquad \mathbf{s}^* = \text{center of the smallest sphere enclosing } \mathbf{s}(t)$$

s(t) is the deviatoric stress; s* is the stabilized residual microstress after elastic shakedown.

$$\max_t \left[\hat{\tau}(t) + \alpha_{DV}\,\sigma_h(t)\right] \le \beta_{DV}, \qquad \hat{\tau} = \frac{\hat{s}_1 - \hat{s}_3}{2}$$

$$\alpha_{DV} = \frac{3t_{-1}}{f_{-1}} - \frac{3}{2}, \qquad \beta_{DV} = t_{-1}, \qquad \text{index} = \frac{\max_t\left(\hat{\tau} + \alpha_{DV}\sigma_h\right)}{\beta_{DV}}$$

Index > 1 predicts finite life (pitting, spalling). Both fatigue limits are reproduced exactly for fully reversed bending and torsion.

The Dang Van diagram plots the path (sigma_h, tau-hat) at the critical depth against the line tau = beta - alpha sigma_h.

Findley life (optional)

The Findley critical-plane damage of the history at each depth (Multiaxial tab k, S-N tab curve) gives the life in passes.

Limits

Elastic, smooth, homogeneous half-spaces; steady rolling; Hertz pressure (no EHL pressure spike); no thermal stress or plasticity.

Point and elliptical contacts are frictionless and on the track centerline. Residual stress is constant with depth.

Contact Fatigue: Elliptical Contact and Ball Bearings (Contact tab)

Contact ellipse (Johnson, Contact Mechanics, 4.2)

$$h = Ax^2 + By^2, \qquad A = \frac{1}{2}\left(\frac{1}{R_{1x}} + \frac{1}{R_{2x}}\right), \qquad B = \frac{1}{2}\left(\frac{1}{R_{1y}} + \frac{1}{R_{2y}}\right)$$

$$\frac{B'}{A'} = \frac{\mathbf{E}(e)/k^2 - \mathbf{K}(e)}{\mathbf{K}(e) - \mathbf{E}(e)}, \qquad k = \frac{d}{c} \le 1, \qquad e^2 = 1 - k^2$$

$$c^3 = \frac{3P\left[\mathbf{K}(e) - \mathbf{E}(e)\right]}{2\pi A' E^* e^2}, \qquad d = kc, \qquad p_0 = \frac{3P}{2\pi cd}$$

A' <= B': A, B sorted; c lies along the smaller curvature. x rolls: a = semi-axis along x, b across. Checked: surface gap = Ax^2 + By^2.

Subsurface stresses (Boussinesq potentials; Johnson eq. 3.18 to 3.20)

$$\psi_1 = \iint p\,\ln(\rho + z)\,dA, \quad \psi = \iint \frac{p}{\rho}\,dA, \quad p = p_0\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$$

$$\sigma_x = \frac{1}{2\pi}\left[2\nu\frac{\partial\psi}{\partial z} - z\frac{\partial^2\psi}{\partial x^2} - (1-2\nu)\frac{\partial^2\psi_1}{\partial x^2}\right], \qquad \sigma_z = \frac{1}{2\pi}\left[\frac{\partial\psi}{\partial z} - z\frac{\partial^2\psi}{\partial z^2}\right]$$

$$\tau_{xy} = -\frac{1}{2\pi}\left[(1-2\nu)\frac{\partial^2\psi_1}{\partial x\,\partial y} + z\frac{\partial^2\psi}{\partial x\,\partial y}\right], \qquad \tau_{xz} = -\frac{z}{2\pi}\frac{\partial^2\psi}{\partial x\,\partial z}, \quad \sigma_y, \tau_{yz} \text{ likewise}$$

Gauss quadrature over x = a sin(alpha), y = b cos(alpha) sin(beta), nodes graded toward the centerline; depths from 0.05 b.

Checked: the circle limit matches Huber to 1e-4 p0; a long ellipse tends to the line contact. Ball bearing: tau_max ~ 0.31 p0.

Ball bearing (Harris, Rolling Bearing Analysis)

$$R_1 = \left[\frac{D}{2}, \frac{D}{2}\right], \qquad R_2 = \left[\pm\frac{d_m \mp D\cos\alpha_0}{2\cos\alpha_0}, -fD\right] \quad (\text{inner}: +, \text{ outer}: -)$$

Contact Fatigue: Lubricant Film (Contact tab)

Elastohydrodynamic film thickness (isothermal, fully flooded, Newtonian)

$$E' = 2E^*, \qquad U = \frac{\eta_0\, u}{E' R_x}, \qquad G = \alpha E', \qquad u = \frac{u_1 + u_2}{2}$$

$$\text{Line (Dowson and Higginson)}: \quad \frac{h_{min}}{R} = 2.65\, U^{0.70} G^{0.54} W^{-0.13}, \qquad W = \frac{w}{E' R}$$

Point and ellipse (Hamrock and Dowson, 1977), k = b / a

$$\frac{h_{min}}{R_x} = 3.63\, U^{0.68} G^{0.49} W^{-0.073} \left(1 - e^{-0.68k}\right), \qquad W = \frac{P}{E' R_x^2}$$

$$\frac{h_c}{R_x} = 2.69\, U^{0.67} G^{0.53} W^{-0.067} \left(1 - 0.61\, e^{-0.73k}\right)$$

Rx: relative radius in the rolling (entraining) direction. The fit covers k = 1 to 8; outside it the app warns.

Specific film thickness

$$\Lambda = \frac{h_{min}}{\sqrt{R_{q1}^2 + R_{q2}^2}}$$

Lambda < 1: boundary lubrication (asperity contact, surface distress, micropitting, wear).

1 to 3: mixed lubrication. > 3: full film, where subsurface (Hertz) fatigue governs.

The film check does not change the stresses: it shows whether the smooth-surface analysis applies. With Lambda < 1, surface-initiated damage and asperity traction (line contact: raise mu) need separate consideration.

eta0: viscosity at the inlet (bulk oil) temperature; alpha: pressure-viscosity coefficient (mineral oil about 10 to 25 1/GPa).

Fatigue Map of a Finite-Element Model (FE Map tab)

Every element is assessed with the single-location methods below; shells at both fibers (Z1, Z2), the worse surface kept.

Static: unit-load stresses x load histories (Superposition method)

$$[\sigma_x,\, \sigma_y,\, \tau_{xy}]_{e,s}\,(t) = \sum_j [\sigma_x,\, \sigma_y,\, \tau_{xy}]^{unit}_{e,s,j}\, L_j\,(t)$$

Unit stresses from SOL 101 (subcase j = load channel j, in subcase order) or a table. Rainflow of the chosen measure (critical plane, max abs principal, signed von Mises) with the S-N tab curve and mean stress correction gives D per block.

Random: stress transfer functions x base PSD (Multiaxial method)

$$\mathbf{G}_\sigma\,(f) = \mathbf{H}^*_{e,s}\,(f)\;\mathbf{H}^T_{e,s}\,(f)\;G_{in}\,(f)\,, \qquad G_{vm}\,(f) = \sum_{a,b} Q_{ab}\,\mathrm{Re}\,G_{ab}\,(f)$$

$$\text{Shells, normal stress}: \quad G_n\,(f,\theta) = \mathbf{c}\,\mathbf{G}_\sigma\,\mathbf{c}^T, \qquad \mathbf{c} = \left[\cos^2\theta,\ \sin^2\theta,\ 2\sin\theta\cos\theta\right]$$

With plane stress, an out-of-plane normal n scales sigma_n by nx^2 + ny^2 <= 1, so the critical plane is normal to the surface: theta = 0 to 179 deg (1 deg); damage at each local maximum of the RMS, the largest kept. Solids search all planes.

Damage per element: the selected spectral method (Dirlik etc.) for the exposure time.

Inputs and outputs

Punch: SORT1 $ELEMENT STRESSES, CQUAD4 (type 33) and CTRIA3 (type 74), centroid. Words after the ID:

real: Z1 sx sy txy angle major minor vm, Z2 ... complex: Z1 sx sy txy (Re/Im or Mag/Phase pairs), Z2 ...

Mesh: GRID with CORD2R/C/S, shells, solids (outer faces drawn) and lines. Export: CSV, legacy VTK (ParaView) cell data.

Seam Weld Fatigue (Seam Weld tab)

Stress history at each weld-toe point = sum of stress per unit load x load history; rainflow (ASTM E1049); Miner sum.

Structural stress (Dong; ASME VIII-2) and the master S-N curve

$$F_i = \frac{l_{i-1}}{6}\left(f_{i-1} + 2f_i\right) + \frac{l_i}{6}\left(2f_i + f_{i+1}\right), \qquad \sigma_m = \frac{f}{t}, \qquad \sigma_b = \frac{6m}{t^2}$$

$$\Delta S_s = \frac{\Delta\sigma_s}{t^{*(2-m)/(2m)}\, I\left(R_b\right)^{1/m}}, \qquad R_b = \frac{|\Delta\sigma_b|}{|\Delta\sigma_m| + |\Delta\sigma_b|}, \qquad m = 3.6, \quad t^* = t/1\,\mathrm{mm}$$

$$I\left(R_b\right)^{1/m} = 2.1549R_b^6 - 5.0422R_b^5 + 4.8002R_b^4 - 2.0694R_b^3 + 0.561R_b^2 + 0.0097R_b + 1.5426$$

$$\Delta S_s = C\,N^{-0.3195}\ (\mathrm{MPa}): \quad C = 19930.2\ (\mathrm{mean}),\ 13875.8\ (-2\,SD),\ 11577.9\ (-3\,SD)$$

Nodal forces F and moments (work equivalence, open weld line) give the line force f and moment m. Each rainflow cycle's membrane and bending ranges give Rb. Constants as tabulated from ASME VIII-2 in Metals 11 (2021) 1952 (code text not checked).

IIW hot-spot and effective notch stress (Hobbacher)

$$\sigma_{hs} = 1.67\,\sigma_{0.4t} - 0.67\,\sigma_{1.0t}, \quad 2.52\,\sigma_{0.4t} - 2.24\,\sigma_{0.9t} + 0.72\,\sigma_{1.4t}, \quad 1.5\,\sigma_{0.5t} - 0.5\,\sigma_{1.5t}$$

$$\text{Type b}: \ 3\,\sigma_{4mm} - 3\,\sigma_{8mm} + \sigma_{12mm}, \qquad 1.5\,\sigma_{5mm} - 0.5\,\sigma_{15mm}$$

$$N = 2\times10^6\left(\frac{FAT\cdot f\left(t\right)}{\Delta\sigma}\right)^3, \qquad f\left(t\right) = \left(\frac{25\,\mathrm{mm}}{t}\right)^n\ \left(t > 25\,\mathrm{mm},\ \text{hot spot}\right)$$

Hot spot FAT 90 or 100; effective notch (1 mm radius) FAT 225. Knee at 1e7 cycles (m2 = 22, Haibach or a limit); IIW f(R).

Ball Bearing Life, ISO 281 (Bearing Life tab)

Rating life (C, C0, Cu, P in lbf or N; the ratios are unit-free)

$$P = F_r \;\; (F_a/F_r \le e)\,, \qquad P = X F_r + Y F_a, \qquad L_{10} = \left(\frac{C}{P}\right)^3 10^6 \,\mathrm{rev}, \qquad L_{10h} = \frac{10^6 L_{10}}{60\, n}$$

$$L_{nm} = a_1\, a_{ISO}\, L_{10}, \qquad a_1 = 1,\ 0.64,\ 0.55,\ 0.47,\ 0.37,\ 0.25 \;\; (R = 90,\ 95,\ 96,\ 97,\ 98,\ 99\,\%)$$

a1 continues to 0.077 at 99.95 %; between table values a1 = 0.95 (ln(100/R)/ln(100/90))^(2/3) + 0.05.

Life modification factor (radial ball; thrust ball uses ec Cu / 3P)

$$a_{ISO} = 0.1 \left[1 - \left(2.5671 - \frac{A}{\kappa^B} \right)^{0.83} \left(\frac{e_C C_u}{P} \right)^{1/3} \right]^{-9.3} \le 50$$

kappa	A	B
0.1 to 0.4	2.2649	0.054381
0.4 to 1	1.9987	0.19087
1 to 4	1.9987	0.071739

$$\kappa = \frac{\nu}{\nu_1}, \qquad \nu_1 = 45000\, n^{-0.83} D_{pw}^{-0.5} \;\; (n < 1000)\,, \qquad \nu_1 = 4500\, n^{-0.5} D_{pw}^{-0.5} \;\; (n \ge 1000)$$

nu, nu1 in mm^2/s (cSt), n in rpm, Dpw in mm (1 in = 25.4 mm). kappa > 4 taken as 4. Compact form: Machines 12 (2024) 444.

Duty cycle, ball loads and contact stress

$$\frac{1}{L_h} = \sum_i \frac{t_i}{L_{h,i}}, \qquad Q_{max} = \frac{F_r}{\cos\alpha \sum_j \cos^{2.5}\psi_j} \approx \frac{4.37\, F_r}{Z \cos\alpha}, \qquad Q = \frac{F_a}{Z \sin\alpha} \;\; (\mathrm{thrust})$$

p0 from the Hertz ellipse on each race (Contact tab equations). C0: 4200 MPa (ISO 76), Cu: 1500 MPa, s0 = C0 / P0,

P0 = max(Fr, 0.6 Fr + 0.5 Fa). Checked: a 6205-size bearing at 5 C0 / Z and 5 Cu / Z gives 4192 and 1468 MPa.

Mission Synthesis: FDS and ERS of a Life Cycle (Mission Synthesis tab)

Base acceleration in G; SDOF absolute response, Q = 1/(2 zeta), h = f / fn; pseudo-damage in G^b (Vibration Fatigue tab basis).

$$|H(h)|^2 = \frac{1 + (h/Q)^2}{(1-h^2)^2 + (h/Q)^2}$$

Sine dwell (frequency f, amplitude A, duration T) and sine sweep (Lalanne Vol. 5, ch. 3)

$$D = f\,T\,(|H|\,A)^b, \qquad E = |H|\,A; \qquad D_{sweep} = n_{pass}\int_{f_1}^{f_2} f\,(|H(f/f_n)|\,A)^b\,\frac{dt}{df}\,df, \qquad E = \max|H|\,A$$

$$\frac{dt}{df} = \frac{60}{R\,f\ln 2}\ \ (\log,\ R\ \text{oct/min}), \qquad \frac{dt}{df} = \frac{60}{R}\ \ (\text{linear},\ R\ \text{Hz/min})$$

Random events: the Vibration Fatigue tab FDS (chosen spectral method) and ERS = sigma sqrt(2 ln(nu0 T)) over the total exposure.

Reference spectra (Lalanne Vol. 5, 11.4.7 and 11.4.8) and the test PSD (11.4.9, eq. [11.11])

$$D_{ref} = \sum_e r_e\,D_e\ \ (\text{sum})\ \ \text{or}\ \ \max_e r_e\,D_e\ \ (\text{envelope}), \qquad E_{ref} = \max_e E_e$$

$$G_{k+1}(f_n) = G_k(f_n)\left[\frac{D_{ref}(f_n)}{D_{test,k}(f_n)}\right]^{2/b}, \qquad G_0 = \frac{2\sigma^2}{\pi f_n Q},\ \ \sigma = \frac{1}{\sqrt{2}}\left(\frac{D_{ref}}{f_n T_{test}\,\Gamma(1+b/2)}\right)^{1/b}$$

The specification is the envelope over b of the test PSDs. ERS check (11.4.10): E_test < E_ref: test too long (shorten);

E_test > E_ref: over-test in peak response (lengthen, or accept). r_e = repeats of event e.

Checked: Lalanne eq. [4.9] to 0.07 %, Example 4.3 ratio 15.33 (book asymptote 16), sweep vs time-domain rainflow 0.5 %.

Fretting Fatigue and Strain-Based Critical Planes (Fretting and Multiaxial tabs)

Cylinder on flat, partial slip with an in-phase bulk stress (Mindlin-Cattaneo; Hills and Nowell)

$$a = \sqrt{\frac{4PR}{\pi E^*}}, \quad p_0 = \frac{2P}{\pi a}, \qquad \frac{c}{a} = \sqrt{1 - \frac{|Q|}{\mu P}}, \qquad \frac{e}{a} = \frac{\sigma_B}{4\mu p_0} \quad (e + c \le a)$$

$$q(x) = \mu p_0 \sqrt{1 - \frac{x^2}{a^2}} - \mu p_0 \frac{c}{a} \sqrt{1 - \frac{(x-e)^2}{c^2}} \quad \text{(stick zone)}, \qquad |q| = \mu p \quad \text{(slip)}$$

Unloading from Qmax reverses the stick zone (Q and sigma_B in phase over the cycle); stresses by superposing the McEwen line solution.

$$\text{Ruiz}: \; F_2 = \max_t \sigma_x \; \max_t (|q|\,\delta), \qquad \delta = \text{slip amplitude}$$

Strain-based critical-plane parameters (elastic strains, Initiation tab constants)

$$\text{SWT}: \; \sigma_{n,max}\,\varepsilon_{n,a} = \frac{\left(\sigma'_f\right)^2}{E}(2N)^{2b} + \sigma'_f\,\varepsilon'_f\,(2N)^{b+c}, \qquad \varepsilon_n = \frac{(1+\nu)\,\sigma_n - \nu\,\mathrm{tr}\,\sigma}{E}$$

$$\text{FS}: \; \gamma_a\left(1 + k_{fs}\frac{\sigma_{n,max}}{S_y}\right) = \frac{\tau'_f}{G}(2N)^{b_0} + \gamma'_f\,(2N)^{c_0}, \qquad \gamma = \frac{\tau}{G}$$

Shear constants estimated from the axial ones when not given: tau_f' = sigma_f'/sqrt(3), gamma_f' = sqrt(3) eps_f', b0 = b, c0 = c.

Sy = cyclic yield K' (0.002)^n' unless entered. sigma_n,max per counted cycle (or over the whole history, the ver 1.3 rule).

Fretting life maps: planes containing y (plane strain), one cycle per point. Multiaxial tab: full 3D plane search.