Vibrationdata FEA: Beams, Plates, Springs, Masses, Rigid Links

Degrees of freedom

Six per node in global axes: $q = [\,T_X,\ T_Y,\ T_Z,\ R_X,\ R_Y,\ R_Z\,]$. Element matrices are built in local axes and rotated: $K = T^T k\, T$, $M = T^T m\, T$, T = block diagonal of the 3×3 direction cosine matrix R (rows = local x, y, z).

Beam: 3D Euler-Bernoulli, 2 nodes, 12 DOF, consistent mass

Local axes

$$\hat{x} = \frac{x_B - x_A}{L}, \quad \hat{z} = \frac{\hat{x} \times v}{|\hat{x} \times v|}, \quad \hat{y} = \hat{z} \times \hat{x} \quad (v = \text{orientation vector; default global } Y)$$

Axial, torsion

$$k = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \ \frac{GJ}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \qquad m = \frac{\mu L}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \ \frac{\rho(I_y + I_z)L}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Bending in x-y (v, θ_z)

$$k = \frac{EI_z}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}, \quad m = \frac{\mu L}{420}\begin{bmatrix} 156 & 22L & 54 & -13L \\ 22L & 4L^2 & 13L & -3L^2 \\ 54 & 13L & 156 & -22L \\ -13L & -3L^2 & -22L & 4L^2 \end{bmatrix}$$

Bending in x-z (w, θ_y) same with I_y and the signs of the L terms reversed $(\theta_y = -dw/dx)$

Mass per length $\mu = \rho A + \text{NSM}$

Plate: flat 4-node shell, 24 DOF (Q4 membrane + MITC4 bending)

Constitutive

$$D_m = \frac{Et}{1-\nu^2}C, \quad D_b = \frac{Et^3}{12(1-\nu^2)}C, \quad C = \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix}, \quad D_s = \frac{5}{6}\,Gt\,I_2$$

Mindlin kinematics

$$\beta_x = \theta_y, \quad \beta_y = -\theta_x, \qquad \kappa = \begin{bmatrix} \beta_{x,x} \\ \beta_{y,y} \\ \beta_{x,y} + \beta_{y,x} \end{bmatrix}, \qquad \gamma = \begin{bmatrix} w_{,x} + \beta_x \\ w_{,y} + \beta_y \end{bmatrix}$$

MITC4 shear (tying)

$$\tilde{\gamma}_{rz} = \frac{1+s}{2}\gamma_{rz}^A + \frac{1-s}{2}\gamma_{rz}^C, \quad \tilde{\gamma}_{sz} = \frac{1+r}{2}\gamma_{sz}^D + \frac{1-r}{2}\gamma_{sz}^B, \quad \begin{bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{bmatrix} = J^{-1}\begin{bmatrix} \tilde{\gamma}_{rz} \\ \tilde{\gamma}_{sz} \end{bmatrix}$$

Tying at the edge midpoints $A = (0,1),\ B = (-1,0),\ C = (0,-1),\ D = (1,0)$.

Stiffness $k = \int_A \left(B_m^T D_m B_m + B_b^T D_b B_b + B_s^T D_s B_s\right) dA$ $(2 \times 2$ Gauss)

Mass $m_{uu} = m_{vv} = m_{ww} = (\rho t + \text{NSM}) \int_A N^T N\, dA, \qquad m_{\theta_x} = m_{\theta_y} = \dfrac{\rho t^3}{12}\displaystyle\int_A N^T N\, dA$

Drilling (θ_z) $k_{\theta_z} = 10^{-3} D_{b,11}\left(I_4 - \dfrac{1}{4}\mathbf{1}\mathbf{1}^T\right)$ (no stiffness for a rigid rotation)

Local frame $\hat{z}$ from the diagonals, $\hat{x}$ along edge 1-2; warped quads projected flat.

Springs, point masses, rigid links, constraints

Spring (global axes)

$$k = \begin{bmatrix} K_s & -K_s \\ -K_s & K_s \end{bmatrix}, \quad K_s = \text{diag}(k_x, k_y, k_z, k_{rx}, k_{ry}, k_{rz}); \ \text{node B} = 0\text{: } K_s \text{ to ground}$$

Point mass $m = \text{diag}(m, m, m, I_{xx}, I_{yy}, I_{zz})$

Rigid link (RBE2) $u_s = u_m + \theta_m \times r, \quad \theta_s = \theta_m, \quad r = x_s - x_m$ (checked slave DOFs only; chains resolved)

Reduction $q = G\, q_i$ (rigid links), constrained DOFs removed, $K_r = G^T K G,\ M_r = G^T M G$

Auto-constrained DOFs with no stiffness and no mass (e.g. rotations of a node with only springs in X)

Massless DOFs $K_c = K_{aa} - K_{ao} K_{oo}^{-1} K_{oa}, \quad q_o = -K_{oo}^{-1} K_{oa}\, q_a$ (exact; null space of K_{oo} dropped)

Distributed loads (BCs/Loads tab; flat plate drilling rotations are constrained automatically)

Beam line load q F_q: $\dfrac{q_x L}{2}$ axial, $\dfrac{qL}{2}$ shear and $\pm\dfrac{qL^2}{12}$ moment at each end; station moments add $\dfrac{qx^2}{2}$ (static

Plate pressure p $F_p = \int_A N^T p\, dA$ on w (quad: 2×2 Gauss; triangle: $pA/3$ per node), + along the normal $n_1 n$

Normal modes and effective mass

Eigenproblem $K\phi = \omega^2 M\phi, \qquad \phi_r^T M\phi_s = \delta_{rs}, \qquad f_n = \dfrac{\omega}{2\pi}$

Participation factor $L_r = \phi_r^T M\iota, \quad \iota$ = rigid-body vector for a unit translation in X, Y or Z

Effective modal mass $m_{\text{eff,r}} = L_r^2, \qquad \sum_r m_{\text{eff,r}}$ = mass that moves (all modes); shown as % of the total mass

Frequency response, modal superposition, damping ratio ζ_r (all modes or per mode)

Modal denominator $D_r(\omega) = \omega_r^2 - \omega^2 + i\,2\zeta\omega_r\omega$

Force at DOF j $H_{ij}(\omega) = \sum_r \dfrac{\phi_{ir}\phi_{jr}}{D_r(\omega)}$ (receptance), mobility $i\omega H_{ij}$, accelerance $-\omega^2 H_{ij}$

Base acceleration $\ddot{u}_b$ $M\ddot{z} + C\dot{z} + Kz = -M\iota\,\ddot{u}_b$ (z = motion relative to the base)

All constrained DOFs and grounded springs move with the base.

Relative displacement $\dfrac{z_i}{\ddot{u}_b} = -\sum_r \dfrac{\phi_{ir}L_r}{D_r(\omega)}$

Acceleration transmissibility $\dfrac{\ddot{u}_i}{\ddot{u}_b} = \iota_i + \omega^2 \sum_r \dfrac{\phi_{ir}L_r}{D_r(\omega)}$ SDOF: $\left|\dfrac{\ddot{u}}{\ddot{u}_b}\right| = \sqrt{\dfrac{1+(2\zeta\rho)^2}{(1-\rho^2)^2+(2\zeta\rho)^2}},\ \rho = \dfrac{f}{f_n}$

Truncation Only computed modes are used: add modes until the effective mass is about 90%.

Units

English: length in, E psi, ρ lbm/in^3, mass lbm, inertia lbm-in^2, k lbf/in, k_r lbf-in/rad, NSM lbm/in or lbm/in^2.
Masses, densities, inertias and NSM are divided by 386 in/s^2; 1 G = 386 in/s^2.

metric: SI (m, N, s, kg, Pa); 1 G = 9.80665 m/s^2. Frequencies in Hz.

Nastran bulk data (Import BDF / Export BDF) and other exports

GRID; MAT1 (E, G, ν, ρ); CBAR + PBAR ($I_1 = I_z$, $I_2 = I_y$, orientation vector written explicitly).
CBEAM/PBEAM and the G0 orientation option are read.
CQUAD4 + PSHELL (MID2 = MID3 = MID, $12I/t^3 = 1$, $t_s/t = 0.833$); CONM2 (offsets not supported).
CBUSH + PBUSH with CID = 0; RBE2; SPC1, SPC and GRID PS → constraints; PARAM,WTMASS applied on import.
Export: SOL 103, PARAM,COUPMASS,1, EIGRL, consistent mass units (lbf-s^2/in or kg), small-field format.
Element IDs are offset by type (see the comments in the file).
ANSYS (Export APDL): BEAM188 (ASEC, cubic, shear stiffness $1000\,AG$, node K on local z), SHELL181,
MASS21, COMBIN14 per spring component, CERIG, D; MODOPT,LANB with consistent mass.
Export M & K: full K, M and reduced $K_r = T^T KT$, $M_r = T^T MT$ with DOF maps (.mat or .csv).

Checks in the test script

Cantilever beam, 20 elements: λL = 1.8751, 4.6941, 7.8548, 10.9955 (error $< 10^{-4}$); effective mass of mode 1 = 0.6131 ρAL.
Simply supported square plate, 16×16: Navier $f_{mn} = \frac{\pi}{2}\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)\sqrt{\frac{D}{\rho t}}$, f_{11} +0.32%; $m_{\text{eff},11} = 64/\pi^4$.
Square plate, $\lambda = \omega a^2\sqrt{\rho t/D}$ ($\nu = 0.3$): clamped 35.985 (+0.9%), free 13.468 (+0.3%, six rigid-body modes).
Rigid links, springs, masses: closed forms and equivalence checks; FRF vs direct solution of $(K - \omega^2 M + i\omega C)$: $< 10^{-13}$.

References

1. J. S. Przemieniecki, *Theory of Matrix Structural Analysis*, McGraw-Hill, 1968 (beam stiffness and consistent mass).
2. K.-J. Bathe, E. N. Dvorkin, "A four-node plate bending element based on Mindlin/Reissner plate theory and a mixed interpolation," *Int. J. Numer. Methods Eng.* 21 (1985) 367-383 (MITC4).
3. A. W. Leissa, *Vibration of Plates*, NASA SP-160, 1969 (square plate frequency parameters).
4. R. D. Blevins, *Formulas for Natural Frequency and Mode Shape*, Van Nostrand Reinhold, 1979.
5. W. C. Young, R. G. Budynas, *Roark's Formulas for Stress and Strain*, 7th ed. (torsion constant of a rectangle).
6. MSC Nastran *Quick Reference Guide* (bulk data entry formats).

Loads, linear static and inertia relief

Load vector $F = F_{\text{nodal}} + M\,\iota\,a_g + F_q + F_p$ (a_g = gravity vector in G times 386 in/s^2 or 9.80665 m/s^2)

Linear static $K_r q_r = T^T F, \quad u = T q_r, \quad$ reactions $r = G^T(Ku - F)$ at the constrained DOFs

Inertia relief $a = (R^T M R)^{-1} R^T F, \qquad Ku = F - MRa, \qquad R^T M u = 0$

$R = 6$ rigid-body vectors about the origin; constraints in the table are ignored

Reported: CG acceleration $a_{cg} = a_O + \alpha \times r_{cg}$ and angular acceleration α

Linear buckling and preload (stress stiffening)

Eigenproblem $(K + \lambda K_G)\,\phi = 0, \quad K_G$ from the static preload; load factor λ: the loads $\times\,\lambda$ buckle

Beam (axial force P) $k_G = \dfrac{P}{30L}\begin{bmatrix} 36 & 3L & -36 & 3L \\ 3L & 4L^2 & -3L & -L^2 \\ -36 & -3L & 36 & -3L \\ 3L & -L^2 & -3L & 4L^2 \end{bmatrix}$ (both planes), torsion $\dfrac{P(I_y + I_z)}{AL}$

Plate (membrane N) $k_G = \int_A G^T \begin{bmatrix} N_x & N_{xy} \\ N_{xy} & N_y \end{bmatrix} G\,dA, \quad G = \begin{bmatrix} N_{,x} \\ N_{,y} \end{bmatrix}$ on w, $N = D_m B_m u$

Checks pinned $\pi^2 EI/L^2$, cantilever $\pi^2 EI/4L^2$ (to 10^{-4}); SS plate $4\pi^2 D/b^2$ (+0.8%)

Preload in the modes $(K + K_G)\,\phi = \omega^2 M \phi$, K_G from the static loads and gravity (tension stiffens)

Also FRF, Guyan, C-B, M & K, BDF (STATSUB), APDL (PERTURB,MODAL)

Checks pinned beam $\omega_n^2 = \omega_{0n}^2\,(1 + P/n^2 P_e)$ to 10^{-5}; SS plate $\omega_{11}^2 = \omega_0^2\,(1 + N_x/N_{cr})$ (-0.13%)

Reduction

Guyan (static) $u = \begin{bmatrix} I \\ -K_{oo}^{-1} K_{oa} \end{bmatrix} u_a, \quad \hat{K} = T^T K T, \ \hat{M} = T^T M T$ (upper-bound frequencies)

Pairs with full modes by MAC $= \dfrac{(\phi_g^T \phi_f)^2}{(\phi_g^T \phi_g)(\phi_f^T \phi_f)}$ on translations

Craig-Bampton $\begin{bmatrix} u_b \\ u_i \end{bmatrix} = \begin{bmatrix} I & 0 \\ \Psi & \Phi \end{bmatrix} \begin{bmatrix} u_b \\ \eta \end{bmatrix}, \quad \Psi = -K_{ii}^{-1} K_{ib}, \ \Phi =$ fixed-interface modes

$K_{cb} = \begin{bmatrix} \hat{K}_{bb} & 0 \\ 0 & \omega^2 \end{bmatrix}, \quad M_{cb} = \begin{bmatrix} \hat{M}_{bb} & M_{b\eta} \\ M_{\eta b} & I \end{bmatrix},$ boundary = constrained DOFs

Interface checks $M_{rb} = R_b^T \hat{M}_{bb} R_b$ (mass, CG), $|\hat{K}_{bb} R_b| \approx 0$, $M_{\eta b} R_b = L$ (participation factors)

Matrices Export M & K: full and reduced K, M; Guyan and C-B saves include T.

Modal truncation and residual vectors (mode-acceleration method)

Force at DOF j $H_{ij}(\omega) = \sum_{r=1}^{n} \dfrac{\phi_{ir}\phi_{jr}}{D_r(\omega)} + \left[(K^{-1})_{ij} - \sum_{r=1}^{n} \dfrac{\phi_{ir}\phi_{jr}}{\omega_r^2}\right]$

Base acceleration $\dfrac{z_i}{\ddot{u}_b} = -\sum_r \dfrac{\phi_{ir} L_r}{D_r(\omega)} + \left[z_{s,i} + \sum_r \dfrac{\phi_{ir} L_r}{\omega_r^2}\right], \quad K z_s = -M\iota$

Use Exact at $\omega = 0$, accurate well below the first omitted mode.

The app reports the % of the static response in the kept modes. Needs constraints.

References (page 3)

7. R. J. Guyan, "Reduction of stiffness and mass matrices," *AIAA J.* 3 (1965) 380.
8. R. R. Craig, M. C. C. Bampton, "Coupling of substructures for dynamic analyses," *AIAA J.* 6 (1968) 1313-1319.
9. R. R. Craig, A. J. Kurdila, *Fundamentals of Structural Dynamics*, 2nd ed., Wiley, 2006 (mode acceleration, residual flexibility).
10. R. D. Cook, D. S. Malkus, M. E. Plesha, R. J. Witt, *Concepts and Applications of Finite Element Analysis*, 4th ed.

Beam end releases, offsets and triangular plates

Pin flags (local DOF r) $u_r = -k_{rr}^{-1} k_{rk}\, u_k, \quad k_c = R_c^T k\, R_c, \;\; m_c = R_c^T m\, R_c$ (released end forces are zero)

Offsets w_A, w_B $u_{end} = u_{node} + \theta \times w$ (beam axis between the offset ends); CBAR PA PB WA WB

Triangle (node 4 = 0) CST membrane + DKT bending: $\beta_s^k = -\dfrac{3}{2L}(w_j - w_i) - \dfrac{1}{4}(\beta_s^i + \beta_s^j), \;\; \beta_n^k = \dfrac{1}{2}(\beta_n^i + \beta_n^j)$

β quadratic, κ linear, 3-point rule; patch tests exact; SS plate: +0.45%

Stress recovery (any displacement vector: static, mode shape, response)

Beam (5 stations) $\sigma = \dfrac{N}{A} + \dfrac{M_y z}{I_y} - \dfrac{M_z y}{I_z}$ at $(\pm c_y, \pm c_z)$, $x/L = 0, \dfrac{1}{4} \ldots 1$; $\tau = \dfrac{Tc}{J}$, $c = \max(c_y, c_z)$

$\sigma_{vm} = \sqrt{\sigma^2 + 3\tau^2}, \;\; \sigma_{1,2} = \dfrac{\sigma}{2} \pm \sqrt{\dfrac{\sigma^2}{4} + \tau^2}$; checked: PLc/I, Tc/J exact

Plate (centre, corners) $\sigma_{top,bot} = \dfrac{N}{t} \pm \dfrac{6M}{t^2}, \quad \sigma_{vm} = \sqrt{\sigma_x^2 - \sigma_x\sigma_y + \sigma_y^2 + 3\tau_{xy}^2}$

Nodal average corner tensors $R^T \sigma R$ averaged at each node (global axes); in-plane $\sigma_{1,2}$, $\tau_{max} = \dfrac{\sigma_1 - \sigma_2}{2}$

Base excitation by modal superposition (ζ_r uniform or per mode, input in G)

Modal coordinate $\ddot{q}_r + 2\zeta_r\omega_r\dot{q}_r + \omega_r^2 q_r = -L_r\, \ddot{u}_b, \quad z = \sum_r \phi_r q_r$ (relative), $\ddot{x} = \iota\ddot{u}_b + \ddot{z}$

Random (PSD P) $H_r(f) = \dfrac{-L_r}{\omega_r^2 - \omega^2 + 2i\zeta\omega_r\omega}, \quad \sigma^2 = \int |H(f)|^2 P(f)\, df$ (log-log PSD, $\Delta f/f \le \zeta/8$)

Plate: $\sigma_{vm,rms}^2 = E[\sigma_x^2 - \sigma_x\sigma_y + \sigma_y^2 + 3\tau_{xy}^2]$ from the cross spectra (Segalman)

Checks: SDOF $\ddot{x}_{rms}^2 = \dfrac{\pi}{2} f_n Q(1 + 4\zeta^2) A_0$ (0.16%), $z_{rms}^2 = \dfrac{A_0}{8\zeta\omega_n^3}$ (0.16%)

Sine one row = steady; sweep = steady-state amplitude envelope; plate σ_{vm}: peak over a cycle

Shock (SRS) $q_r = \dfrac{L_r\, SRS(f_r)}{\omega_r^2}$; combine modes: SRSS, ABS, NRL = max $+\sqrt{\sum \text{rest}^2}$; SDOF exact

Transient $x_{k+1} = e^{F\Delta t} x_k + \Gamma_0 \ddot{u}_{b,k} + \Gamma_1(\ddot{u}_{b,k+1} - \ddot{u}_{b,k})/\Delta t$ (exact for straight lines)

from $\exp\left(\begin{bmatrix} F & g & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Delta t \right)$; SDOF half-sine vs ode45: 2.5×10^{-10} in

Participation and modal mass (Modes tab, Table list)

Participation $\Gamma = \phi^T M R_p$ (TX ... RZ); R_p about the CG, the origin or a point; $m_{eff} = \Gamma^2$

Modal mass (max = 1) $\psi = \phi/c$, c = largest translation: $m = 1/c^2$, $\Gamma_\psi = c\,\Gamma$; cantilever mode 1: 0.25 m, 1.566

Picking and view

Click a node, beam or plate on the plot to select its table row; Delete removes it (a node only when nothing uses it).
Zoom In / Out, Fit, Reset, Pan, Rotate, Box Zoom; External Fig; Contour: von Mises, principal or max shear.
Show panel: hide element types or show one group. Undo (30 steps), Paste rows from Excel, Refine plates 1 to 4.
Save Curve (random, sine, transient): two columns, time (sec) or frequency (Hz) and the node response or element stress.

References (page 4)

11. J.-L. Batoz, K.-J. Bathe, L.-W. Ho, "A study of three-node triangular plate bending elements," *IJNME* 15 (1980) 1771-1812.
12. D. J. Segalman, C. W. G. Fulcher, G. M. Reese, R. V. Field, "An efficient method for calculating RMS von Mises stress in a random vibration environment," *J. Sound Vib.* 230 (2000) 393-410.
13. N. C. Nigam, P. C. Jennings, "Calculation of response spectra from strong-motion earthquake records," *BSSA* 59 (1969) 909-922.
14. T. Irvine, Vibrationdata tutorials: Miles equation, SRS, modal combination (NRL).