Critical Speed Hand Estimates: Rayleigh, Dunkerley, Closed Form and FE Check

critical_speed_estimates_a ver 1.0 by Tom Irvine

Notation

m_i disc mass at x_i, y_i static deflection at disc i under 1 g, g gravity, EI bending stiffness, $\bar{m} = \rho A$ mass/length, L shaft length, a bearing span, k_1, k_2 bearing stiffnesses (rigid: $1/k = 0$), ω rad/s, $N_c = \frac{30}{\pi}\,\omega$ rpm.

Methods

1. Rayleigh, static deflection $\quad \omega^2 = \dfrac{g\sum m_i y_i}{\sum m_i y_i^2}, \qquad N_c = \dfrac{30}{\pi}\sqrt{\dfrac{g\sum m_i y_i}{\sum m_i y_i^2}} \qquad$ (upper bound)

   with shaft mass $\quad \omega^2 = \dfrac{\mathbf{y}^T\mathbf{F}}{\mathbf{y}^T\mathbf{M}\,\mathbf{y}}, \qquad \mathbf{K}\mathbf{y} = \mathbf{F} = g\,\mathbf{M}\,\mathbf{r} \quad (\mathbf{r} = 1 \text{ on lateral dof})$

   one shape iteration $\quad \mathbf{K}\mathbf{y}_2 = \mathbf{M}\mathbf{y}, \qquad \omega^2 = \dfrac{\mathbf{y}_2^T\mathbf{M}\,\mathbf{y}}{\mathbf{y}_2^T\mathbf{M}\,\mathbf{y}_2} \qquad$ (upper bound, closer to FE)

2. Dunkerley $\quad \dfrac{1}{\omega^2} = \dfrac{1}{\omega_s^2} + \sum_i \dfrac{1}{\omega_i^2}, \qquad \omega_i^2 = \dfrac{1}{m_i\,\alpha_{ii}} \qquad$ (lower bound)

   ω_s = shaft alone on its bearings, α_{ii} = influence coefficient at disc i

3. Closed form, one disc $\quad \omega = \dfrac{1}{\sqrt{m\,\alpha}} \qquad$ (uniform massless shaft)

   between bearings $\quad \alpha = \dfrac{x^2b^2}{3EIs} + \left(\dfrac{b}{s}\right)^2\dfrac{1}{k_1} + \left(\dfrac{x}{s}\right)^2\dfrac{1}{k_2}, \quad b = s - x,\ s = L \text{ or } a$

   centre disc, rigid $\quad \alpha = \dfrac{L^3}{48EI}, \qquad$ with shaft mass: $m + \dfrac{17}{35}\,\bar{m}L$

   overhung disc $\quad \alpha = \dfrac{c^2(a+c)}{3EI} + \left(\dfrac{c}{a}\right)^2\dfrac{1}{k_1} + \left(\dfrac{a+c}{a}\right)^2\dfrac{1}{k_2}, \quad c = x - a$

   cantilever $\quad \alpha = \dfrac{x^3}{3EI}, \qquad$ tip disc with shaft mass: $m + \dfrac{33}{140}\,\bar{m}L$

   shaft alone, uniform $\quad \omega = \left(\dfrac{\pi}{L}\right)^2\sqrt{\dfrac{EI}{\bar{m}}}$ (simply supported), $\qquad \left(\dfrac{1.8751}{L}\right)^2\sqrt{\dfrac{EI}{\bar{m}}}$ (cantilever)

4. FE beam check $\quad \left(\mathbf{K} - \omega^2\mathbf{M}\right)\phi = 0 \quad$ Euler-Bernoulli elements, 2 dof per node, about 120 elements

Element stiffness $\dfrac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}$, consistent mass $\dfrac{\bar{m}l}{420}\begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix}$

Discs add m_i to the lateral dof at x_i; flexible bearings add k_j to the lateral dof; rigid bearings and the clamp remove dof. Percent difference $= 100\,(N_c/N_{c,FE} - 1)$. Rayleigh $\geq$ FE $\geq$ Dunkerley for the same model.

Assumptions

- Lateral bending in one plane, Euler-Bernoulli beam (no shear deformation), solid or hollow circular sections.
- Discs are point masses: no rotary or polar inertia, so no gyroscopic stiffening. No damping.
- Critical speed = first non-rotating natural frequency. Bearings act as linear lateral springs, no moment stiffness.
- Rayleigh uses the deflection under gravity; for overhung rotors this trial shape can be poor (estimate high).

Units

English: lengths and diameters in, E psi, ρ lbm/in^3, m lbm, k lbf/in.
Masses and densities are divided by 386.088 in/s^2 (lbf-s^2/in), $g = 386.088$ in/s^2.

metric: lengths and diameters m, E GPa, ρ kg/m^3, m kg, k N/m, $g = 9.80665$ m/s^2.
Steel: $E = 30$e6 psi $= 206.8$ GPa, $\rho = 0.283$ lbm/in$^3 = 7833$ kg/m^3.

References: W. T. Thomson, *Theory of Vibration with Applications* (Rayleigh and Dunkerley); S. S. Rao, *Mechanical Vibrations*; J. M. Vance, *Rotordynamics of Turbomachinery*; R. D. Blevins, *Formulas for Natural Frequency and Mode Shape* (beam constants).
The 17/35 and 33/140 effective shaft masses are the classical Rayleigh values; the app checks them against the FE model (within 1%).