Reciprocating Engine Shaking Forces: Free Inertia Forces and Moments

engine_shaking_forces_a ver 1.0 by Tom Irvine

Notation and axes

r crank radius, l connecting rod length, $\lambda = r/l$, $\omega = 2\pi N/60$, N speed (rpm), s cylinder spacing.
x along the crankshaft (cylinder 1 at $x = 0$), y horizontal, z vertical. Throw i is at crank angle $\theta + \phi_i$, measured from $+z$ toward $+y$; cylinder i axis is at α_i from $+z$ (inline 0, V bank A $+V/2$, B $-V/2$, flat $\pm 90°$).
m_{rec} reciprocating mass per cylinder (piston, pin, rings, small-end share), m_{rot} rotating mass per crank throw.

Slider-crank kinematics (cylinder i, local angle $t = \theta + \phi_i - \alpha_i$ from TDC)

1. Piston position — $x_p = r\cos t + \sqrt{l^2 - r^2\sin^2 t}$

2. Exact acceleration — $\ddot{x}_p = -r\omega^2\left[\cos t + \dfrac{\lambda\cos 2t + \lambda^3\sin^4 t}{\left(1 - \lambda^2\sin^2 t\right)^{3/2}}\right]$

3. Series (to 4th order) — $\ddot{x}_p \approx -r\omega^2\left[\cos t + A_2\cos 2t - A_4\cos 4t\right]$

4. Series coefficients — $A_2 = \lambda + \dfrac{\lambda^3}{4} + \dfrac{15\lambda^5}{128}$, $\quad A_4 = \dfrac{\lambda^3}{4} + \dfrac{3\lambda^5}{16}$ $\quad (A_6 = \dfrac{9\lambda^5}{128}$ omitted)

5. Inertia force on the frame — $F_i = -m_{rec}\,\ddot{x}_p = m_{rec}\,r\,\omega^2\,g(t)$, along the cylinder axis, outward positive

Summing over cylinders and throws

6. Reciprocating forces — $F_z = \sum_i F_i\cos\alpha_i$, $\quad F_y = \sum_i F_i\sin\alpha_i$

7. Counterweight per throw — $m_{cw} = \dfrac{c_{rot}}{100}\,m_{rot} + \dfrac{c_{rec}}{100}\,m_{rec}$ (at radius r, opposite the throw)

8. Rotating force per throw — $(m_{rot} - m_{cw})\,r\,\omega^2\,[\sin(\theta+\phi_k),\ \cos(\theta+\phi_k)]$ in (y, z)

9. Free moments — $M_y = \sum(x - x_c)\,F_z$ (pitch), $\quad M_z = \sum(x - x_c)\,F_y$ (yaw), $\quad x_c = \overline{x}_i$

10. Order n amplitude (series) — $\left|\sum_i c_n\,m_{rec}r\omega^2\,e^{\,jn(\phi_i-\alpha_i)}\,(\cos\alpha_i,\ \sin\alpha_i,\ \ldots)\right|$, $\quad c_1 = 1,\ c_2 = A_2,\ c_4 = -A_4$

11. Order n amplitude (exact) — $|C_n|$, $\quad C_n = \dfrac{2}{N_t}\displaystyle\sum_k F(\theta_k)\,e^{-jn\theta_k}$ (FFT of one revolution, $N_t = 1440$)

An order is *balanced* when its force and moment amplitudes vanish. A 1st-order quantity that *rotates with the crank* (F_y phasor $= -j\,F_z$ phasor) can be cancelled by crank counterweights; a reciprocating (planar) one needs balance shafts.
Firing: cylinder i is at TDC when $\theta = \alpha_i - \phi_i$ (mod 360°); the firing order sets the firing intervals only.

Classic results (series model, $m = m_{rec}$, about the engine centre)

Single cylinder: 1st $F_z = mr\omega^2$, 2nd $F_z = A_2\,mr\omega^2$; with $c_{rec} = 50\%$ the 1st order splits $\frac{1}{2}$ vertical, $\frac{1}{2}$ horizontal.
I3 $(0, 120, 240°)$: 1st and 2nd forces balanced; free pitch moments $\sqrt{3}\,s\,mr\omega^2$ (1st) and $\sqrt{3}\,s\,A_2\,mr\omega^2$ (2nd).
I4 $(0, 180, 180, 0°)$: 1st order and all moments balanced; free 2nd-order force $4A_2\,mr\omega^2 \approx 4\lambda\,mr\omega^2$ (vertical).
I6 $(0, 240, 120, 120, 240, 0°)$: 1st, 2nd and 4th forces and moments all balanced.
V8 cross-plane (90° V, $0, 90, 270, 180°$): forces and 2nd moments balanced; 1st-order couple $\sqrt{10}\,s\,(m_{rec} + m_{rot} - m_{cw})\,r\omega^2$ rotating with the crank, cancelled by counterweights with $m_{cw} = m_{rot} + m_{rec}$ per throw ($c_{rot} = c_{rec} = 100\%$).
V8 flat-plane: 2nd-order horizontal force $4\sqrt{2}\,A_2\,mr\omega^2$. Boxer 4: 2nd-order yaw couple $2\,s\,A_2\,mr\omega^2$. V12 60°, boxer 6: balanced.

Units

English: r, l, s in, masses lbm (divided by 386.088 in/s^2), forces lbf, moments lbf-in, speed rpm.
metric: r, l, s mm (converted to m), masses kg, forces N, moments N-m, speed rpm.

Assumptions

Rigid crank and frame, constant speed, no piston-pin offset. Gas-pressure torque and its reaction are out of scope.
Connecting rod represented by two point masses (big end in m_{rot}, small end in m_{rec}); its rotary inertia couple is neglected.
Cylinders sharing a crank pin are placed in the same axial plane (rod offset neglected). Valve gear and accessories are not included.

References: C. F. Taylor, *The Internal Combustion Engine in Theory and Practice*, Vol. 2, MIT Press, ch. 8 (balance); J. B. Heywood, *Internal Combustion Engine Fundamentals*, sec. 2.2 (kinematics); C. E. Wilson and J. P. Sadler, *Kinematics and Dynamics of Machinery*, engine balancing chapter; Wikipedia, "Piston motion equations" and "Crossplane". The classic results above were also confirmed numerically by the app's test script (series vs exact vs FFT). Book chapter numbers are from memory.