# FE Rotor Critical Speeds: Flexible Shaft with Disks on Bearings



## Finite element model

Beam elements, 4 DOF per node: $q = [\,x,\ \psi_x,\ y,\ \psi_y\,]$, $\psi$ = section rotation in the $xz$ and $yz$ planes.
Euler-Bernoulli: $\psi = dx/dz$. Spin $\Omega$ about $+z$. Each shaft segment is split into equal elements.
Stations are the segment ends; disks and bearings sit at stations.

Equation of motion

$$M\ddot{q} + (C + \Omega G)\,\dot{q} + Kq = F$$

Shear parameter

$$\Phi = \frac{12EI}{\kappa G_s A L^2}, \qquad G_s = \frac{E}{2(1+\nu)}$$

Shear coefficient (Cowper)

$$\kappa = \frac{6(1+\nu)(1+m^2)^2}{(7+6\nu)(1+m^2)^2 + (20+12\nu)m^2}, \qquad m = \frac{D_i}{D_o}$$

Euler-Bernoulli option

$\Phi = 0$ (rotary inertia and gyroscopic terms kept, shear deformation dropped)

Element stiffness

$$K_e = \frac{EI}{(1+\Phi)L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & (4+\Phi)L^2 & -6L & (2-\Phi)L^2 \\ -12 & -6L & 12 & -6L \\ 6L & (2-\Phi)L^2 & -6L & (4+\Phi)L^2 \end{bmatrix}$$

Translational mass

$$M_T = \frac{\rho AL}{(1+\Phi)^2}\begin{bmatrix} m_1 & m_2 & m_3 & m_4 \\ m_2 & m_5 & -m_4 & m_6 \\ m_3 & -m_4 & m_1 & -m_2 \\ m_4 & m_6 & -m_2 & m_5 \end{bmatrix}$$

$$m_1 = \frac{13}{35} + \frac{7\Phi}{10} + \frac{\Phi^2}{3}, \qquad m_2 = \left(\frac{11}{210} + \frac{11\Phi}{120} + \frac{\Phi^2}{24}\right)L$$

$$m_3 = \frac{9}{70} + \frac{3\Phi}{10} + \frac{\Phi^2}{6}, \qquad m_4 = -\left(\frac{13}{420} + \frac{3\Phi}{40} + \frac{\Phi^2}{24}\right)L$$

$$m_5 = \left(\frac{1}{105} + \frac{\Phi}{60} + \frac{\Phi^2}{120}\right)L^2, \qquad m_6 = -\left(\frac{1}{140} + \frac{\Phi}{60} + \frac{\Phi^2}{120}\right)L^2$$

Rotary inertia mass

$$M_R = \frac{\rho I}{(1+\Phi)^2 L}\begin{bmatrix} 6/5 & r_2 & -6/5 & r_2 \\ r_2 & r_3 & -r_2 & r_4 \\ -6/5 & -r_2 & 6/5 & -r_2 \\ r_2 & r_4 & -r_2 & r_3 \end{bmatrix}$$

$$r_2 = \left(\frac{1}{10} - \frac{\Phi}{2}\right)L, \quad r_3 = \left(\frac{2}{15} + \frac{\Phi}{6} + \frac{\Phi^2}{3}\right)L^2, \quad r_4 = \left(-\frac{1}{30} - \frac{\Phi}{6} + \frac{\Phi^2}{6}\right)L^2$$

Element mass

$M_e = M_T + M_R$ (same in the $xz$ and $yz$ planes; consistent mass)

Gyroscopic (shaft)

$G_{xy} = 2M_R, \quad G_{yx} = -2M_R$ (polar inertia = 2× diametral, circular section)

Rigid disk at a node

$M_{xx} \mathrel{+}= m, \quad M_{\psi\psi} \mathrel{+}= I_d$ (both planes), $\quad G_{\psi_x\psi_y} \mathrel{+}= I_p, \quad G_{\psi_y\psi_x} \mathrel{+}= -I_p$

Bearing at a node

$K_{xx} \mathrel{+}= k_{xx}, \quad K_{yy} \mathrel{+}= k_{yy}, \quad C_{xx} \mathrel{+}= c_{xx}, \quad C_{yy} \mathrel{+}= c_{yy}$ (no cross-coupled terms)

Thin uniform disk

$$I_p = \frac{m}{8}(D_o^2 + D_i^2), \qquad I_d = \frac{I_p}{2} + \frac{mt^2}{12}$$ ($t$ = disk thickness)

## Assumptions

Linear, small motion about the spinning axis; constant speed for each eigenvalue solution.
Circular solid or hollow shaft sections; rigid disks; linear bearings without cross-coupled terms.
Lateral motion only (no axial or torsional DOF); shaft internal (rotating) damping neglected.
Massless, undamped DOFs (for example a massless shaft) are removed by static condensation.

## Whirl frequencies, damping and critical speeds

State-space eigenvalues $\begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}(C+\Omega G) \end{bmatrix} z = \lambda z, \qquad \lambda = -\zeta\omega_n \pm i\,\omega_d$

Frequency, damping ratio $f = \dfrac{\omega_d}{2\pi}$ (Hz), $\qquad \zeta = \dfrac{-\mathrm{Re}\,\lambda}{|\lambda|}$

Log decrement $\delta = \dfrac{-2\pi\,\mathrm{Re}\,\lambda}{\mathrm{Im}\,\lambda} = \dfrac{2\pi\zeta}{\sqrt{1-\zeta^2}}$ $\qquad (\delta < 0$ is unstable)

Forward / backward $\sum_{\text{nodes}} \mathrm{Im}(\bar{X}\,Y) < 0 \;\Rightarrow$ forward (orbit $x \to y$, same sense as the spin)

Critical speed $f_j(\Omega) = n\,\dfrac{N}{60}, \qquad n = 1$ (and 2), $\qquad N =$ shaft speed (rpm)

Crossings found on the Campbell sweep, refined by root finding.

Undamped map (per plane) $\left[K_1 + k\sum_b e_b e_b^T - \omega^2(M_1 - G_1)\right] Q = 0, \qquad N_{cr} = \dfrac{60\,\omega}{2\pi}$

1X forward synchronous whirl, same isotropic $k$ at every bearing, no damping.

Rigid supports Translations fixed at the bearing stations (pinned): the high-$k$ limit of the map.

Unbalance force $F_x = me\,\Omega^2, \qquad F_y = -i\,me\,\Omega^2$ $\qquad$ (rotating with the shaft)

Unbalance response $\left[K - \Omega^2 M + i\,\Omega\,(C + \Omega G)\right] Q = F$

Orbit size $a = \dfrac{1}{2}\left(|X + iY| + |X - iY|\right)$ $\qquad$ (major semi-axis, zero-to-peak)

## Closed forms used in the test script

Pinned shaft, Euler-Bernoulli $\omega_n = \left(\dfrac{n\pi}{L}\right)^2 \sqrt{\dfrac{EI}{\rho A}}$

with rotary inertia $\omega^2 = \dfrac{EIk^4}{\rho A + \rho I k^2}, \qquad k = \dfrac{n\pi}{L}$

Timoshenko $\dfrac{\rho^2 I}{\kappa G_s}\,\omega^4 - \left[\rho A + \rho I k^2\left(1 + \dfrac{E}{\kappa G_s}\right)\right]\omega^2 + EIk^4 = 0$

Jeffcott rotor $\omega = \sqrt{\dfrac{k_s}{m}}, \qquad \dfrac{1}{k_s} = \dfrac{L^3}{48EI}\ \left(+\dfrac{L}{4\kappa G_s A}\ \text{with shear}\right)$

Benchmark: Friswell et al. Example 5.9.1, 1.5 m steel shaft, 50 mm dia., disks 280 and 350 mm dia. × 70 mm, bearings 1 MN/m, 6 Timoshenko elements: 0 rpm 86.66, 274.31, 716.78 rad/s; 4000 rpm 85.39, 87.80, 251.78, 294.71, 600.18 rad/s (values from the ROSS validation page; the app agrees to the printed digits).

## Units

**English**: length in, $E$ psi, $\rho$ lbm/in$^3$, mass lbm, $I_p, I_d$ lbm-in$^2$, $k$ lbf/in, $c$ lbf-s/in, $me$ lbm-in.
Masses, $\rho$, inertias and $me$ are divided by 386.088 in/s$^2$. Unbalance response in mils (0.001 in).

**metric**: length m, $E$ Pa, $\rho$ kg/m$^3$, mass kg, $I_p, I_d$ kg-m$^2$, $k$ N/m, $c$ N-s/m, $me$ kg-m.
Unbalance response in $\mu$m. Both systems: speeds in rpm, frequencies in Hz.

## References


1. M. I. Friswell, J. E. T. Penny, S. D. Garvey, A. W. Lees, *Dynamics of Rotating Machines*, Cambridge University Press, 2010, Chapter 5 (finite element rotor models), Example 5.9.1. Published values checked via the ROSS validation page (ross.readthedocs.io, "Validation"); the book itself was not consulted.
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3. H. D. Nelson, "A finite rotating shaft element using Timoshenko beam theory,' *J. Mech. Des.* 102 (1980) 793-803.
4. G. R. Cowper, "The shear coefficient in Timoshenko's beam theory,' *J. Appl. Mech.* 33 (1966) 335-340.
5. J. S. Przemieniecki, *Theory of Matrix Structural Analysis*, McGraw-Hill, 1968 (beam element with shear deformation).