Spectral Kurtosis & Kurtogram: Best Band for Envelope Analysis

kurtogram_a ver 1.0 by Tom Irvine

Spectral kurtosis by STFT (Antoni 2006)

1. STFT, Hann window $X_w(t_i, f) = \sum_{n=0}^{N_w-1} w_n\, x(n + iN_w/2)\, e^{-j2\pi f n \Delta t}$, $i = 1 \ldots K$ frames (50 % overlap)

2. Spectral kurtosis $SK(f) = \dfrac{K+1}{K}\, \dfrac{\langle |X_w(t_i,f)|^4 \rangle_i}{\langle |X_w(t_i,f)|^2 \rangle_i^2} - 2$

3. Reference values Gaussian noise: $SK = 0 \pm \dfrac{2}{\sqrt{K}}$, 99 % limit $2.576\, \dfrac{2}{\sqrt{K}}$; tone: $SK = -1 + \dfrac{1}{K}$

The factor $(K+1)/K$ makes the estimate unbiased for K independent frames of Gaussian noise ($E[\cdot] = 2K/(K+1)$). SK is not shown within f_s/N_w of 0 Hz and Nyquist. Short windows resolve short transients, long windows narrow bands.

Fast Kurtogram (Antoni 2007): 1/3-binary tree of quasi-analytic filters

4. Prototype low-pass $h = \text{fir1}(16,\, 0.4)$, $g = \text{fir1}(24,\, 0.4 \cdot 2/3)$ (Hamming-windowed sinc, unit DC gain)

5. Binary split ($\downarrow 2$) $c_{k+1,\,2i-1} = \left[(h\, e^{j\pi n/4}) * c_{k,i} \right]_{\downarrow 2}$, $c_{k+1,\,2i} = (-1)^n \left[(h\, e^{j3\pi n/4}) * c_{k,i} \right]_{\downarrow 2}$

6. 1/3 split ($\downarrow 3$) $c_{k+\log_2 3,\, 3(i-1)+m} = \left[(g\, e^{j(2m-1)\pi n/6}) * c_{k,i} \right]_{\downarrow 3}$, $m = 1, 2, 3$

7. Level, width, centre $k = 0,\ 1,\ \log_2 3,\ 2,\ 1 + \log_2 3,\ 3, \ldots, k_{max}$; $B_w = \dfrac{f_s}{2^{k+1}}$, $f_c = \left(i - \dfrac{1}{2} \right) B_w$

8. Band kurtosis $K(f_c, B_w) = \dfrac{\langle |c_{k,i}|^4 \rangle}{\langle |c_{k,i}|^2 \rangle^2} - 2$ (complex envelope, mean removed); level 0 (real x): -3

9. Best band $(f_c, B_w) = \arg\max K$ $(B_w \geq B_{min})$; $f_1 = f_c - B_w/2$, $f_2 = f_c + B_w/2$

$c_{0,1} = x - \bar{x}$. The first 16 (binary) or 24 (1/3) filtered samples are skipped. Kurtogram rows: $2k_{max}$, columns: $3 \cdot 2^{k_{max}}$.
Maximum level $k_{max} \leq \text{floor}(\log_2 N) - 6$. Gaussian noise: $K \approx 0 \pm 2/\sqrt{n_b}$, n_b = band samples.
Bands touching 0 Hz or Nyquist read high for noise (the complex envelope there is not circular).

Envelope analysis of the best band

10. Analytic band-pass $Z_k = 2X_k$ for $f_1 \leq f_k \leq f_2$, $Z_k = 0$ otherwise; $z = \text{IFFT}(Z)$, $x_f = \text{Re}\, z$

11. Hilbert envelope $e(t) = |z(t)|$

12. Envelope spectrum $E(f_k) = \dfrac{2\left| \sum_n w_n (e_n - \bar{e})\, e^{-j2\pi k n/N} \right|}{\sum_n w_n}$, w_n Hann, $\Delta f = \dfrac{f_s}{N}$

Same scaling as envelope_fft / envelope_fft_a (6th-order Butterworth band-pass there): enter f_1, f_2 as its Lower / Upper Freq.
Check: $A(1 + m\cos 2\pi f_m t)\cos 2\pi f_c t$ gives $e = A(1 + m\cos 2\pi f_m t)$ and $E(f_m) = A\, m$.

Bearing defect frequencies (outer race fixed, no slip)

13. Geometry ratio $r = \dfrac{d}{D}\cos\theta$, $f_r = \dfrac{\text{rpm}}{60}$, n = number of rolling elements

14. Defect frequencies $\text{BPFO} = \dfrac{n f_r}{2}(1 - r)$, $\text{BPFI} = \dfrac{n f_r}{2}(1 + r)$, $\text{BSF} = \dfrac{D}{2d} f_r (1 - r^2)$, $\text{FTF} = \dfrac{f_r}{2}(1 - r)$

d rolling element diameter, D pitch diameter, θ contact angle. A rolling-element spall can also show at 2 BSF. Real bearings slip, so measured lines are often 1–2 % off.

Units: **English**: acceleration G, diameters in. **metric**: acceleration m/s^2, diameters mm. Time s, frequency Hz, speed rpm. Kurtosis is dimensionless; only d/D enters the bearing formulas, so the band and defect frequencies are unit-independent.

Assumptions: constant speed, uniform time step, mean removed, ideal band-pass for the envelope; no order tracking.

References: J. Antoni, "The spectral kurtosis: a useful tool for characterising non-stationary signals," *MSSP* 20(2), 2006, 282–307.
J. Antoni and R. B. Randall, "The spectral kurtosis: application to the vibratory surveillance and diagnostics of rotating machines," *MSSP* 20(2), 2006, 308–331.
J. Antoni, "Fast computation of the kurtogram for the detection of transient faults," *MSSP* 21(1), 2007, 108–124.
R. B. Randall and J. Antoni, "Rolling element bearing diagnostics – a tutorial," *MSSP* 25(2), 2011, 485–520.
Filter-bank constants (eqs. 4–6) match Antoni's Fast_Kurtogram code and MATLAB `kurtogram`; not checked against the full text of the 2007 paper. Bearing formulas: standard kinematics, as in Randall and Antoni (2011).