Machine on Isolators, Six dof: Rigid Machine on N Discrete Isolators

Coordinates and notation

Axes: x fore-aft, y lateral, z vertical up. Isolator locations p_i, CG c and points are entered in the same axes.
Six degrees of freedom at the CG: $q = \{\, u_x \;\; u_y \;\; u_z \;\; \theta_x \;\; \theta_y \;\; \theta_z \,\}^T$ (roll θ_x, pitch θ_y, yaw θ_z).
Isolator i: stiffness $k_{x,i}, k_{y,i}, k_{z,i}$ along the axes, optional rotational stiffness $k_{rx,i}, k_{ry,i}, k_{rz,i}$, $i = 1 \ldots N$.

Mass and stiffness matrices

1. Mass matrix $\quad M = \begin{bmatrix} m\,I_3 & 0 \\ 0 & J \end{bmatrix}, \quad J = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix}$

2. Products of inertia $\quad I_{xy} = \int (x - x_c)(y - y_c)\, dm$, etc., about the CG

3. Isolator offset $\quad r_i = p_i - c, \qquad u_i = T_i\, q, \qquad T_i = [\; I_3 \quad -S(r_i) \;]$

4. Cross-product matrix $\quad S(r) = \begin{bmatrix} 0 & -r_z & r_y \\ r_z & 0 & -r_x \\ -r_y & r_x & 0 \end{bmatrix}, \qquad S(r)\, a = r \times a$

5. Stiffness matrix $\quad K = \sum_{i=1}^{N} T_i^T\, k_i\, T_i \;+\; \sum_{i=1}^{N} \begin{bmatrix} 0 & 0 \\ 0 & k_{r,i} \end{bmatrix}$

6. Isolator stiffness $\quad k_i = \mathrm{diag}(k_x, k_y, k_z)_i, \qquad k_{r,i} = \mathrm{diag}(k_{rx}, k_{ry}, k_{rz})_i$

7. Rotation-free isolators $\quad k_{r,i} = 0$ (point springs: each isolator transmits force only)

Natural frequencies and mode shapes

8. Eigenproblem $\quad K\,\phi_r = \omega_r^2\, M\,\phi_r, \qquad f_r = \dfrac{\omega_r}{2\pi}, \qquad \phi_r^T M \phi_r = 1, \qquad r = 1 \ldots 6$

9. Kinetic energy fractions $\quad e_{jr} = \phi_{jr}\,(M\phi_r)_j, \qquad \sum_{j=1}^{6} e_{jr} = 1$

10. Mode description $\quad$ largest e_{jr} names the motion (fore-aft, lateral, bounce, roll, pitch, yaw)

11. Coupling $\quad$ coupled if a second $e_{jr} \geq 5\%$; the table lists all six e_{jr} in percent

Symmetric isolators in the plane of the CG decouple the six motions, e.g. bounce $\omega = \sqrt{\sum k_z / m}$. A CG height h above the isolator plane couples fore-aft with pitch and lateral with roll (the rocking modes).

Damping (same for every isolator)

12. Viscous, damping ratio ζ $\quad c_i = \dfrac{2\zeta}{\omega_{ref}}\, k_i, \quad \omega_{ref} = \sqrt{\dfrac{\sum k_z}{m}} \quad \Rightarrow \quad s(\omega) = 1 + i\,\dfrac{2\zeta\omega}{\omega_{ref}}, \quad \zeta_r = \zeta\,\dfrac{\omega_r}{\omega_{ref}}$

13. Hysteretic, loss factor η $\quad s(\omega) = 1 + i\,\eta \qquad$ (complex stiffness $k(1 + i\eta)$)

14. Switching type $\quad \eta = 2\zeta$ (equal damping at resonance)

The viscous dampers are stiffness proportional, so the modes stay real; the vertical 1-DOF case has exactly $c = 2\zeta\sqrt{km}$.

Units

English: m lbm and I lbm-in^2 (both $\div$ 386.088), lengths in, k lbf/in, k_r lbf-in/rad, U oz-in ($\div$ 16 $\div$ 386.088), F lbf, $g = 386.088$ in/s^2, displacement in, acceleration G.

metric: m kg, I kg-m^2, lengths mm (m internally), k N/m, k_r N-m/rad, U g-mm ($\times 10^{-6}$ kg-m), F N, $g = 9.80665$ m/s^2, displacement mm, acceleration G. Frequencies Hz, speed rpm.

Excitation at 1X and 2X

15. Running speed $\Omega = 2\pi N/60, \qquad f_{1X} = N/60, \qquad f_{2X} = 2 f_{1X}$

16. Rotating unbalance $F = U\,\Omega^2\,(e_1 - i\,e_2) \;\Rightarrow\; U\Omega^2\,(e_1\cos\Omega t + e_2 \sin\Omega t), \quad e_1 \perp a,\; e_2 = a \times e_1$

17. Directed force $F = F_0\,a, \qquad a =$ unit shaft axis or force direction

18. 2X component $|F^{(2X)}| = \dfrac{\alpha}{100}\,|F^{(1X)}|$ at 2Ω, same pattern and point

19. Generalized force $Q = \left\{ \begin{array}{c} F \\ (p_F - c) \times F \end{array} \right\}$

Forced response

20. Response $\left[\, s(\Omega)\,K - \Omega^2 M \,\right] X = Q$

21. Point P $u_P = T_P\,X, \qquad a_P = -\Omega^2\,u_P$ (reported in G: $|a_P|/g$)

22. Isolator force (on foundation) $F_i = s(\Omega)\,k_i\,T_i\,X, \qquad F_T = \sum_{i=1}^{N} F_i$

23. Peak of a complex vector $\hat{V} = \max_t \left|\mathrm{Re}\left(V e^{i\Omega t}\right)\right| = \sqrt{\dfrac{1}{2}\left(\|V\|^2 + |V^T V|\right)}$

24. Transmissibility $TR = \hat{F}_T/\hat{F}, \qquad$ isolation efficiency $= 100\,(1 - TR)\ \%$

25. Static limit check $\Omega \to 0:\;\; F_T \to F$ (force rows of $KX = Q$)

26. Static load (gravity) $K q_s = \{0\ \ 0\ \ -mg\ \ 0\ \ 0\ \ 0\}^T, \quad P_i = -k_{z,i}\,u_{z,i}, \quad \sum P_i = mg$

The sweep plot repeats the response over frequency (unbalance force grows as Ω^2, a directed force is constant).

1-DOF vertical design check (Crede, Harris)

27. Static deflection, frequency $\delta_{st} = \dfrac{mg}{\sum k_z}, \qquad f_n = \dfrac{1}{2\pi}\sqrt{\dfrac{\sum k_z}{m}} = \dfrac{1}{2\pi}\sqrt{\dfrac{g}{\delta_{st}}}, \qquad \rho = \dfrac{f_{1X}}{f_n}$

28. Transmissibility $TR = \sqrt{\dfrac{1 + (2\zeta\rho)^2}{(1-\rho^2)^2 + (2\zeta\rho)^2}}$ (viscous), $\qquad TR = \sqrt{\dfrac{1+\eta^2}{(1-\rho^2)^2 + \eta^2}}$ (hysteretic)

29. Required for efficiency E $TR_t = 1 - \dfrac{E}{100}, \qquad \rho = \sqrt{1 + \dfrac{1}{TR_t}}, \qquad f_{n,req} = \dfrac{f_{1X}}{\rho}$

30. Required stiffness $\sum k_{z,req} = m\,(2\pi f_{n,req})^2, \qquad \delta_{req} = \dfrac{g}{(2\pi f_{n,req})^2}$

The required values use the undamped formula $TR = 1/(\rho^2 - 1),\;\; \rho > \sqrt{2}$.

Assumptions

Rigid machine, rigid foundation, small linear motion. Isolator axes parallel to x, y, z (no inclined isolators).
Gravity preload does not change the stiffness. Isolators with only k_z leave the horizontal motions unrestrained (rejected).
Isolator moments (rotational stiffness) are not included in the transmitted force F_T.

References

1. C. E. Crede and J. E. Ruzicka, Theory of Vibration Isolation, Chapter 30 in C. M. Harris and A. G. Piersol (eds.), *Harris' Shock and Vibration Handbook*, 5th ed., McGraw-Hill, 2002 (chapter number per the 5th edition).
2. C. E. Crede, *Vibration and Shock Isolation*, Wiley, 1951.
3. E. I. Rivin, *Passive Vibration Isolation*, ASME Press, 2003.
4. J. P. Den Hartog, *Mechanical Vibrations*, 4th ed., McGraw-Hill, 1956 (Dover reprint), transmissibility.