Shaft Fatigue at a Section: Goodman, Gerber, ASME Elliptic, Soderberg

Notation

Solid round rotating shaft, diameter d at the section; shoulder diameter D, fillet or notch root radius r. M_a, M_m bending, T_a, T_m torque, F_a, F_m axial force (alternating, mean). A steady transverse load on a rotating shaft gives fully reversed bending, so it is entered as M_a; M_m is only a moment that rotates with the shaft. S_{ut}, S_y ultimate and yield strengths.

1. Stress concentration and notch sensitivity

1. Fatigue factors $K_f = 1 + q\,(K_t - 1), \qquad K_{fs} = 1 + q_s\,(K_{ts} - 1), \qquad K_{f,ax} = 1 + q\,(K_{t,ax} - 1)$

2. Neuber / Peterson q $q = \dfrac{1}{1 + \sqrt{a}/\sqrt{r}}$ (r in in, $\sqrt{a}$ in $\sqrt{\text{in}}$, S_{ut} in ksi, valid 50 to 250 ksi)

3. Bending, axial $\sqrt{a} = 0.246 - 3.08(10^{-3})S_{ut} + 1.51(10^{-5})S_{ut}^2 - 2.67(10^{-8})S_{ut}^3$

4. Torsion $\sqrt{a} = 0.190 - 2.51(10^{-3})S_{ut} + 1.35(10^{-5})S_{ut}^2 - 2.67(10^{-8})S_{ut}^3$

5. Shoulder fillet K_t $K_t = C_1 + C_2 x + C_3 x^2 + C_4 x^3, \quad x = \dfrac{2h}{D}, \quad h = \dfrac{D - d}{2}, \quad C_i = c_{i1} + c_{i2}\sqrt{h/r} + c_{i3}\,h/r$

Peterson curve fits as given by Pilkey, *Formulas for Stress, Strain and Structural Matrices*, 2nd ed.: bending and axial $0.1 \le h/r \le 2$ and $2 \le h/r \le 20$ (two coefficient sets), torsion $0.25 \le h/r \le 4$. Outside these ranges the fit is extrapolated and the app says so.

Shigley Table 7-1 first-iteration estimates (K_t bending, K_{ts} torsion, K_t axial)

Shoulder fillet, sharp ($r/d = 0.02$)	2.7	2.2	3.0
Shoulder fillet, well rounded ($r/d = 0.1$)	1.7	1.5	1.9
End-mill keyseat ($r/d = 0.02$)	2.14	3.0	2.14*
Sled-runner keyseat	1.7	3.0*	1.7*
Retaining ring groove	5.0	3.0	5.0

* not in Table 7-1; assumed by the app (axial = bending, sled-runner torsion = end-mill value). Or select Manual K_t and q.

2. Endurance limit (Marin equation)

6. Corrected limit $S_e = k_a\,k_b\,k_c\,k_d\,k_e\,k_f\,S'_e, \qquad S'_e = 0.5\,S_{ut}$ for $S_{ut} \le 200$ ksi (1400 MPa), else 100 ksi (700 MPa)

7. Surface k_a $k_a = a\,S_{ut}^b \le 1$

Finish	a (ksi)	a (MPa)	b
Ground	1.34	1.58	−0.085
Machined or cold-drawn	2.70	4.51	−0.265
Hot-rolled	14.4	57.7	−0.718
As-forged	39.9	272	−0.995
Polished (test specimen)	$k_a = 1$		

8. Size k_b, English $k_b = 0.879\,d^{-0.107}$ (0.11 to 2 in), $0.91\,d^{-0.157}$ (2 to 10 in) (bending, torsion)

9. Size k_b, metric $k_b = 1.24\,d^{-0.107}$ (2.79 to 51 mm), $1.51\,d^{-0.157}$ (51 to 254 mm)

10. Load k_c $k_c = 1$ for combined loading; the alternating axial stress is divided by 0.85 (Shigley Sec. 6-14)

11. Temperature k_d $k_d = 0.975 + 0.432(10^{-3})T_F - 0.115(10^{-5})T_F^2 + 0.104(10^{-8})T_F^3 - 0.595(10^{-12})T_F^4, \;\; 70 \le T_F \le 1000\,^\circ\text{F}$

12. Reliability k_e $k_e = 1 - 0.08\,z_a, \quad z_a = \sqrt{2}\,\text{erf}^{-1}(2R - 1)$ (90 %: 0.897, 99 %: 0.814, 99.9 %: 0.753)

13. Miscellaneous k_f user input (corrosion, plating, fretting, ...), default 1

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3. Von Mises stresses at the notch

14. Alternating
$$\sigma'_a = \left[\left(K_f \frac{32 M_a}{\pi d^3} + \frac{K_{f,ax}}{0.85} \frac{4 F_a}{\pi d^2} \right)^2 + 3 \left(K_{fs} \frac{16 T_a}{\pi d^3} \right)^2 \right]^{1/2}$$

15. Mean
$$\sigma'_m = \left[\left(K_f \frac{32 M_m}{\pi d^3} + K_{f,ax} \frac{4 F_m}{\pi d^2} \right)^2 + 3 \left(K_{fs} \frac{16 T_m}{\pi d^3} \right)^2 \right]^{1/2}$$

16. Maximum (yield)
$$\sigma'_{max} = \left[\left(\sigma_{a,b} + \sigma_{m,b} + \sigma_{a,ax} + \sigma_{m,ax} \right)^2 + 3 \left(\tau_a + \tau_m \right)^2 \right]^{1/2}$$ (Shigley Eq. 7-15 plus axial)

K_f is applied to the mean stresses too (conservative). Magnitudes are added at the critical fiber, so a compressive F_m gets no credit.

4. Fatigue and yield factors of safety

17. Modified Goodman
$$\frac{1}{n} = \frac{\sigma'_a}{S_e} + \frac{\sigma'_m}{S_{ut}}$$

18. Gerber
$$n = \frac{1}{2} \left(\frac{S_{ut}}{\sigma'_m} \right)^2 \frac{\sigma'_a}{S_e} \left[-1 + \sqrt{1 + \left(\frac{2 \sigma'_m S_e}{S_{ut} \sigma'_a} \right)^2} \right]$$

19. ASME elliptic
$$n = \left[\left(\frac{\sigma'_a}{S_e} \right)^2 + \left(\frac{\sigma'_m}{S_y} \right)^2 \right]^{-1/2}$$

20. Soderberg
$$\frac{1}{n} = \frac{\sigma'_a}{S_e} + \frac{\sigma'_m}{S_y}$$

21. First-cycle yield
$$n_y = \frac{S_y}{\sigma'_{max}}, \qquad \text{Langer (conservative):} \quad n_y = \frac{S_y}{\sigma'_a + \sigma'_m}$$

Goodman and Gerber do not guard against yielding, so check n_y. The fatigue diagram plots each criterion, the Langer line, the load line through the origin and the load point; markers show where the load line meets each criterion $(n\,\sigma'_m, n\,\sigma'_a)$.

5. Minimum diameter for a target factor of safety n

22. DE-Goodman (7-8)
$$d = \left\{ \frac{16 n}{\pi} \left(\frac{1}{S_e} \left[4 (K_f M_a)^2 + 3 (K_{fs} T_a)^2 \right]^{1/2} + \frac{1}{S_{ut}} \left[4 (K_f M_m)^2 + 3 (K_{fs} T_m)^2 \right]^{1/2} \right) \right\}^{1/3}$$

23. DE-ASME ellip. (7-12)
$$d = \left\{ \frac{16 n}{\pi} \left[4 \left(\frac{K_f M_a}{S_e} \right)^2 + 3 \left(\frac{K_{fs} T_a}{S_e} \right)^2 + 4 \left(\frac{K_f M_m}{S_y} \right)^2 + 3 \left(\frac{K_{fs} T_m}{S_y} \right)^2 \right]^{1/2} \right\}^{1/3}$$

k_b and q depend on d, so the equations are iterated with D/d and r/d held fixed. With an axial force the app instead solves $n(d) = n_{target}$ numerically with the full model of Sections 3 and 4. The ASME elliptic form is the basis of the former ASME B106.1M.

Units

English: d, D, r in, M, T lbf-in, F lbf, S and σ ksi, temperature °F.
metric: d, D, r mm, M, T N-m (times 1000 to N-mm), F N, S and σ MPa, °C ($T_F = 1.8\,T_C + 32$). q uses S_{ut} in ksi, r in in. The metric k_a and k_b constants are Shigley's own, so metric and English results agree to about 0.1 %.

References and source status

R. G. Budynas and J. K. Nisbett, *Shigley's Mechanical Engineering Design*, McGraw-Hill, 9th ed. (2011) and 10th ed. (2015), Ch. 6 (Eq. 6-18 to 6-35, Table 6-2, Sec. 6-12 to 6-14) and Ch. 7 (Eq. 7-7 to 7-16, Table 7-1). Check case: Example 7-1 (9th ed.). The 11th ed. (2020) revised Table 6-2 (e.g. machined $a = 2.00$ ksi, $b = -0.217$); this app uses the 9th/10th-edition table.

W. D. Pilkey, *Formulas for Stress, Strain and Structural Matrices*, 2nd ed., Wiley (2005): Peterson shoulder-fillet K_t fits.

ASME B106.1M-1985, *Design of Transmission Shafting*: listed as inactive / withdrawn (withdrawal reported as 1994); not confirmed from ASME directly. Shown for history: the DE-ASME elliptic criterion generalizes its design equation.

Not confirmed from a primary source: Table 7-1 values other than the keyseat and sharp-fillet rows, the k_d polynomial coefficients, and the Example 7-1 fatigue factors of safety. Confirmed from course notes reproducing Example 7-1: K_f, K_{fs}, q, q_s, $k_b, k_e, \sigma'_a, \sigma'_m$ and $n_y = 4.50$. Other formulas were checked against public course notes; check the textbook.