Torsional Forced Response: Lumped Inertia-Shaft Chain

torsional_forced_response_a ver 1.0 by Tom Irvine

Model

Rigid inertias J_i with damping c_i to ground, joined by massless shafts k_e, c_e (shaft e joins stations e and $e+1$).
Free-free chain, linear, steady-state harmonic response at each shaft speed N (rpm).

1. Matrices — $\mathbf{M} = \mathrm{diag}(J_i)$, $\mathbf{K} = \sum_e k_e \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}_e$, $\mathbf{C} = \mathrm{diag}(c_i) + \sum_e c_e \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}_e$
2. Natural frequencies — $\left[\mathbf{K} - \omega^2\mathbf{M}\right]\{\phi\} = \mathbf{0}$, $f_n = \omega/2\pi$ (rigid-body mode $\omega = 0$ dropped)
3. Excitation, order h — $T_s(t) = A\cos(h\,\Omega t + \psi) = \mathrm{Re}\{Ae^{i\psi}e^{ih\Omega t}\}$, $\Omega = 2\pi N/60$
4. Steady state — $\left[\mathbf{K} - \omega_h^2\mathbf{M} + i\,\omega_h\mathbf{C}\right]\{\Theta_h\} = \{T_h\}$, $\omega_h = h\,\Omega$
5. Shaft torque — $T_{e,h} = (k_e + i\,\omega_h c_e)\,(\Theta_{e+1,h} - \Theta_{e,h})$
6. Shear stress — $\tau = \dfrac{16\,T}{\pi d^3}$ (solid round shaft; no stress concentration)
7. Interference speed — $N_{cr} = \dfrac{60\,f_n}{h}$ rpm (Campbell diagram: lines $f = hN/60$ cross f_n)

Combining the orders (alternating value = half of peak-to-peak)

Peak of synthesized waveform: $x(\theta) = \mathrm{Re}\sum_h X_h\,e^{ih\theta}$ over one full cycle $\theta = 0 \ldots 2\pi p$ (p = revolutions, 2 for half orders), $x_{alt} = \frac{1}{2}(\max x - \min x)$, peaks refined by a parabola. *Sum of order amplitudes*: $x_{alt} = \sum_h |X_h|$ (upper bound). One order: $x_{alt} = |X_h|$. The speed grid is refined to 0.1% steps within ±2% of each N_{cr}.

Reciprocating engine option

Cylinder j at station s_j fires at crank angle α_j (after cylinder 1). Harmonic torque coefficient $T_{n,h}$ (pressure units) and phase ψ_h:

$$T_{j,h} = T_{n,h}\,\frac{\pi B^2}{4}\,R\,e^{\,i(\psi_h - h\,\alpha_j)} \qquad B = \text{bore},\ \ R = \text{crank radius (stroke/2)}$$

Even firing: the k-th cylinder in the firing order gets $\alpha = (k-1)\,720°/n$ (4-stroke) or $(k-1)\,360°/n$ (2-stroke).
Orders: multiples of 1/2 (4-stroke), integers (2-stroke). Relative vector sum $|\sum_j e^{-ih\alpha_j}|/n = 1$ marks a major order.
Engine rows are added to the excitation table.
$T_{n,h}$ must come from engine data or the literature; the values in the example are illustrative only.

Units

English: J lbf-in-s^2 (= lbm-in^2/386.088), k lbf-in/rad, c lbf-in-s/rad, A, T lbf-in, d, B, R in, τ, T_n psi.
metric: J kg-m^2, k N-m/rad, c N-m-s/rad, A, T N-m, d, B, R mm, τ, T_n MPa: $\tau = 16000\,T/(\pi d^3)$, $T_{j,h} = 10^{-3}\,T_n\frac{\pi}{4}B^2R$.
Speeds in rpm, frequencies in Hz, angles in degrees, phases in degrees in both systems.

Assumptions and limits

Linear, viscous damping; vibratory torque only (the mean torque is not added); no gear backlash or nonlinear couplings.
Single shaft speed. For a geared train enter values referred to one shaft (Jn^2, kn^2); the actual torque in a shaft turning n times the reference speed is the referred torque divided by n (the app reports referred torque).
The allowable alternating shear stress is a user input: take it from the applicable rules or standard (not built in).

Checks (test_tfr.m, R2024a)

2 inertias: $f_n = \frac{1}{2\pi}\sqrt{k(J_1+J_2)/(J_1J_2)}$, undamped $T_s = A\,\frac{J_2}{J_1+J_2}\,/\,(1-r^2)$, damped closed form, resonance at $60f_n/h$.
Undamped direct solution = modal superposition (rigid-body mode included); metric = English; f_n = torsional_critical_speeds_a.

References

J. P. Den Hartog, *Mechanical Vibrations*, McGraw-Hill (Dover reprint): torsional vibration and engine harmonics.
W. Ker Wilson, *Practical Solution of Torsional Vibration Problems*, Chapman & Hall.
E. J. Nestorides (ed.), *A Handbook on Torsional Vibration*, B.I.C.E.R.A., Cambridge University Press, 1958.