Unbalance Response: Finite Element Rotor with Disks & Damped Bearings

unbalance_response_a ver 1.0 by Tom Irvine

Rotor model

Same model as campbell_diagram_a: isotropic rotor, complex coordinates $w = x + i\,y$, slope θ, Euler-Bernoulli shaft elements with rotary inertia and gyroscopics, rigid disks, isotropic bearings with viscous damping.

1. Element stiffness

$$\mathbf{K}_e = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

2. Translational mass

$$\mathbf{M}_T = \frac{\rho AL}{420}\begin{bmatrix} 156 & 22L & 54 & -13L \\ 22L & 4L^2 & 13L & -3L^2 \\ 54 & 13L & 156 & -22L \\ -13L & -3L^2 & -22L & 4L^2 \end{bmatrix}$$

3. Rotary inertia, gyroscopics

$$\mathbf{M}_R = \frac{\rho I}{30L}\begin{bmatrix} 36 & 3L & -36 & 3L \\ 3L & 4L^2 & -3L & -L^2 \\ -36 & -3L & 36 & -3L \\ 3L & -L^2 & -3L & 4L^2 \end{bmatrix} \qquad \mathbf{G}_e = 2\,\mathbf{M}_R$$

4. Disk at node j — M_{w_j} += m M_{θ_j} += I_d G_{θ_j} += I_p

5. Bearing at node j — K_{w_j} += k C_{w_j} += c K_{θ_j} += k_r C_{θ_j} += c_r

6. Unbalance at node j — $U = m_u\,e$ (mass × radius) at angle ϕ: $F_{w_j} = U\,\Omega^2 e^{\,i(\Omega t+\phi)}$

Synchronous response

7. Equation of motion, spin speed Ω — $\mathbf{M}\ddot{\mathbf{q}} + (\mathbf{C} - i\,\Omega\,\mathbf{G})\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{F}$

8. Forward response, $\mathbf{q} = \mathbf{Q}e^{i\Omega t}$ — $\left[\mathbf{K} - \Omega^2(\mathbf{M} - \mathbf{G}) + i\,\Omega\,\mathbf{C}\right]\mathbf{Q} = \Omega^2\,\mathbf{U}e^{i\phi}$

9. Displacement and phase — $\delta_{pp} = 2\,|Q_w|$ phase lag $= -\arg Q_w$ (behind the $\phi = 0$ reference)

10. Bearing force — $|F_b| = |(k + i\,\Omega\,c)\,Q_w|$

11. Critical speeds, damping ignored — $\left[\mathbf{K} - \Omega^2(\mathbf{M} - \mathbf{G})\right]\mathbf{Q} = \mathbf{0}$ (forward 1X, as in campbell_diagram_a)

12. Check: Jeffcott rotor — $r = \dfrac{U\Omega^2}{k_s - m\Omega^2 + i\,c\,\Omega}$ $k_s = \dfrac{48EI}{L^3}$ (disk at midspan, pinned ends)

Amplification factor and separation margin (operating range N_{min} to N_{max})

$AF = \dfrac{N_c}{N_2 - N_1}$ half-power points N_1, N_2 at 0.707× peak. One point only: $AF = N_c/(2|N_c - N_{1,2}|)$, marked *.

$AF < 2.5$: critically damped, no separation margin required.

Peak below the range: $SM_{req} = \min\left[16,\ 17\left(1 - \frac{1}{AF-1.5}\right)\right]$ %, $SM = 100\,\dfrac{N_{min} - N_c}{N_{min}}$ %

Peak above the range: $SM_{req} = \min\left[26,\ 10 + 17\left(1 - \frac{1}{AF-1.5}\right)\right]$ %, $SM = 100\,\dfrac{N_c - N_{max}}{N_{max}}$ %

Margins follow API Standard 617 and API RP 684 (lateral analysis). Verify against the applicable edition.

Units

English: E psi, ρ lbm/in^3, lengths in, m lbm, I_p, I_d lbm-in^2, k lbf/in, c lbf-s/in, k_r lbf-in/rad, c_r lbf-in-s/rad, U lbm-in. Mass terms and U are divided by 386.088 in/s^2. Displacement in mils pk-pk, force in lbf.

metric: E Pa, ρ kg/m^3, lengths m, m kg, I_p, I_d kg-m^2, k N/m, c N-s/m, k_r N-m/rad, c_r N-m-s/rad, U kg-m. Displacement in micrometers pk-pk, force in N. Speeds in rpm in both systems.

References: H. D. Nelson and J. M. McVaugh, *The Dynamics of Rotor-Bearing Systems Using Finite Elements*, ASME Journal of Engineering for Industry, 98(2), 1976, pp. 593-600.

API Standard 617, *Axial and Centrifugal Compressors and Expander-compressors*; API RP 684, *Rotordynamic Tutorial.*